The mean number of sites visited by a random walk pinned at a distant point

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Abstract
This paper concerns the number \( Z_n \) of sites visited up to time \( n \) by a random walk \( S_n \) having zero mean and moving on the two dimensional square lattice \( \mathbb{Z}^2 \). Asymptotic evaluation of the conditional expectation of \( Z_n \) for large \( n \) given that \( S_n = x \) is carried out under some exponential moment condition. It gives an explicit form of the leading term valid uniformly in \( (x, n) \), \(|x| < cn\).

Keywords: Range of random walk; pinned random walk; Cramér transform; local central limit theorem.

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1 Introduction and main results
This paper is a continuation of the paper [12] by the present author, where the expectation of the cardinality of the range of a pinned random walk is studied when the random walk of prescribed length is pinned at a point within a parabola of space-time variables. In this paper we deal with the case when it is outside a parabola at which the walk is pinned and compute the asymptotic form of the (conditional) expectation. To this end we derive a local limit theorem valid outside parabolas by using Cramér transform.

The random number, denoted by \( Z_n \), of the distinct sites visited by a random walk in the first \( n \) steps is one of typical characteristics or functionals of the random walk paths. The expectation of \( Z_n \) may be regarded as the total heat emitted from a site at the origin which is kept at the unit temperature. The study of \( Z_n \) is traced back to Dvoretzky and Erdős [2] in which the law of large numbers of \( Z_n \) is obtained for simple random walk. Nice exposition of their investigation and an extension of it is found in [10]. For the pinned walk the expectation of \( Z_n \) is computed by [12], [4]. Corresponding problems for Brownian sausage have also been investigated (often earlier) (cf. [11], [3] for free motions and [6], [7], [14] for bridges).

Let \( S_n = X_1 + \cdots + X_n \) be a random walk on the two-dimensional square lattice \( \mathbb{Z}^2 \) starting at the origin. Here the increments \( X_j \) are i.i.d. random variables defined on some probability space \((\Omega, \mathcal{F}, P)\) taking values in \( \mathbb{Z}^2 \). The random walk is supposed to be irreducible and having zero mean: \( E[X] = 0 \). Here and in what follows we write \( X \) for a random variable having the same law as \( X_1 \).

For \( \lambda \in \mathbb{R}^2 \), put
\[
\phi(\lambda) = \log E[e^{\lambda \cdot X}]
\]

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The mean number of sites visited

and for \( \mu \in \mathbb{R}^2 \) let \( m(\mu) \) be the value of \( \lambda \) determined by

\[
\nabla \phi(\lambda) \bigg|_{\lambda=m(\mu)} = \mu :
\]

(1.1)

\( m(\mu) \) is well defined if \( \mu \) is an interior point of the image set \( \nabla \phi(\Xi) \) of

\[ \Xi = \{ \lambda : E[|X|e^{\lambda X}] < \infty \}. \]

Since \( \nabla \phi(0) = 0 \), if the interior of \( \Xi \) contains the origin, then so does the interior of \( \nabla(\Xi) \).

Let \( f_0(n) \) be the probability that the walk returns to the origin for the first time at the \( n \)-th step \((n \geq 1)\) and define

\[
H(\mu) = \sum_{k=1}^{\infty} f_0(k) \left( 1 - e^{-k\phi(m(\mu))} \right).
\]

Let \( Z_n \) \((n = 1, 2, \ldots)\) denote the cardinality of the set of sites visited by the walk up to time \( n \), namely

\[ Z_n = \sharp\{S_1, S_2, \ldots, S_n\}. \]

Let \( Q \) be the covariance matrix of \( X \) and \(|Q|\) be the determinant of \( Q \).

**Theorem 1.** Suppose that \( \phi(\lambda) < \infty \) in a neighborhood of the origin and let \( K \) be a compact set contained in the interior of \( \Xi \). Then,

\[
H(\mu) = \frac{\pi \sqrt{|Q|}}{-\log |\mu|} + O \left( \frac{1}{(\log |\mu|)^2} \right) \text{ as } |\mu| \to 0,
\]

(1.2)

and, uniformly for \( x \in \mathbb{Z}^2 \) satisfying \( x/n \in \nabla \phi(K) \) and \(|x| \geq \sqrt{n}\),

\[
E\left[ Z_n \mid S_n = x \right] = nH(x/n) + O \left( \frac{1}{(\log n) \vee (\log |x/n|)^2} \right) \text{ as } n \to \infty.
\]

(1.3)

**Example 1.** For symmetric simple random walk we have \( e^{\phi(\lambda)} = \frac{1}{2} \cosh \alpha + \frac{1}{2} \cosh \beta \) for \( \lambda = (\alpha, \beta) \). Given \( x/n = \mu + o(1) \), the leading term \( nH(x/n) \) in (1.3) may be computed from

\[
H(\mu) = 1 - \sum_{j=1}^{\infty} e^{f_0(2j)} 2^{2j} (\cosh \alpha + \cosh \beta)^{2j}, \quad \mu = \nabla \phi(\lambda) = \frac{(\sinh \alpha, \sinh \beta)}{\cosh \alpha + \cosh \beta}.
\]

The derivative of \( H \) along a circle centered at the origin directed counter-clockwise is given by

\[
\nabla H(\mu) \cdot \mu^\perp = C_0(\mu) \mu_1 \mu_2 (\mu_2^2 - \mu_1^2),
\]

where \( \mu^\perp = (\mu_2, -\mu_1) \) and \( C_0(\mu) \) is a smooth positive function of \( \mu \neq 0 \). (See Appendix (B).)

We see shortly that the behavior of the probability \( P_h[S_n = x] \) differs greatly in different directions of \( x \) as soon as \(|x|/n^{3/4} \) gets large even if \( Q \) is isotropic. (See Proposition 2 below.) According to Theorem 1.2, in contrast to this, the leading term of \( E[Z_n \mid S_n = x] \) as \( x/n \to 0 \) as well as that of \( H(\mu) \) as \( \mu \to 0 \) is rotation invariant; only when \(|x|/n \) is bounded away from zero, \( E[Z_n \mid S_n = x] \) in general becomes dependent on directions of \( x \).

The case \(|x| = O(\sqrt{n})\) is studied in [12] under certain mild moment conditions. If we assume the rather strong moment condition \( E[|X|^4] < \infty \), the result is presented as
The mean number of sites visited

follows: for each $a_0 > 0$ it holds that uniformly for $|x| < a_0 \sqrt{n}$, as $n \to \infty$

$$E[Z_n|S_n = x] = 2\pi \sqrt{|Q|} n \int_{e^{-n}}^\infty W(u) du + \frac{4\sqrt{|Q|} x^2}{(\log n)^2} \left( \log \frac{n}{|x|^2} + O(1) \right) + o(1) + b_3 O(|x|) \left( \frac{1}{\log n} \right),$$

(1.4)

where $W(\lambda) = \int_0^\infty ([\log t]^2 + \pi^2)^{-1} e^{-\lambda t} dt$ and $\tilde{x} = Q^{-1/2} x$. We have the asymptotic expansion $\int_\lambda^\infty W(u) du = (\log \lambda)^{-1} - \gamma (\log \lambda)^{-2} + (\gamma^2 - \frac{1}{2} \pi^2) (\log \lambda)^{-3} + \cdots$ ($\lambda \to \infty$), where $\gamma = 0.5772\ldots$ (Euler’s constant).

Brownian analogue of (1.4) is given in [14], the proof being similar but rather more involved than for the random walk case.

Remark 1. By a standard argument we have

$$1 - \sum_1^\infty e^{-k\lambda} f_0(k) = \left( \frac{1}{2\pi} \right)^2 \int_{[-\pi,\pi]^2} \frac{d\theta}{1 - e^{-\lambda} E[e^{\theta X}]}^{-1} \quad (\lambda > 0).$$

Substitution from $\lambda^\infty E[e^{\theta X}] = e^{\phi(\theta)}$ and $\lambda = \phi(m(\mu))$ therefore yields

$$\frac{1}{H(\mu)} = \frac{1}{2\pi} \int_{[-\pi,\pi]^2} \frac{d\theta}{1 - \exp(-\phi(m(\mu)) + \phi(i\theta))} \quad (\mu \neq 0).$$

(1.5)

Remark 2. For $d \geq 3$ the results analogous to (1.4) are obtained by the same method. Here only a result of [12] for the case $d = 3$ is given:

Suppose $d = 3$ and $E[|X|^4] < \infty$. Then uniformly for $|x| < a_0 \sqrt{n}$, as $n \to \infty$

$$E[Z_n|S_n = x] = q_0 n + \frac{q_0 |x|}{2\pi \sqrt{|Q|}} + O\left( \frac{1}{1 + |x|} \right) + b_3 O(1) + o \left( \frac{1}{\sqrt{n}} \right),$$

where $q_0 = P[S_n \neq 0$ for all $n \geq 1].$

Remark 3. For random walks of continuous time parameter the asymptotic form of the expectation are deduced from those of the embedded discrete time walks by virtue of the well-known purely analytic result as given in [5].

For the proof of Theorem 1 we derive a local limit theorem, an asymptotic evaluation of the probability $P[S_n = x]$, denoted by $q^n(x)$, for large $n$, that is sharp uniformly for the space-time region $\sqrt{n} \leq |x| < \epsilon n$ with $\epsilon > 0$ (Lemma 3). As a byproduct of it we obtain the following proposition which lucidly exhibits what happens for variables $\sqrt{n} < |x| < n$ with $n$ large: if all the third moments vanish, then the ratio of the probabilities $q^n(x)$ among directions of $x$ with the same modulus $|x|$ can be unbounded as $|x|/n^{3/4}$ gets large; if not, this may occur as $|x|/n^{2/3}$ gets large. This result though not directly used in the proof of Theorem 1 is interesting by itself.

**Proposition 2.** Uniformly in $x$, as $n \to \infty$ and $|x|/n \to 0$,

$$q^n(x) = \frac{\nu 1(q^n(x) \neq 0) e^{-x Q^{-1} x/2n} \left( 1 + O\left( \frac{|x| + 1}{n} \right) \right)}{2\pi n \sigma^2} \times \exp \left\{ \kappa_3 \left( \frac{x}{n} \right) + \kappa_4 \left( \frac{x}{n} \right) + O\left( \frac{|x|^4}{n^4} \right) \right\},$$

where $\kappa_3(\mu) = \frac{1}{6} E[(Q^{-1} X \cdot \mu)^3]$ and $\kappa_4$ is a homogeneous polynomial of degree 4. If all the third moments of $X$ vanish, then

$$\kappa_4(\mu) = -\frac{1}{8} (Q^{-1}(\mu))^2 + \frac{1}{24} E[(Q^{-1} X \cdot \mu)^4].$$

The mean number of sites visited

Example 2. For the same simple random walk as in Example 1 it follows from Proposition 2 that

\[ q^n(x) = \frac{4e^{-|x|^2/2}}{\pi n} \left( 1 + O\left( \frac{|x| + 1}{n} \right) \right) \exp \left\{ - \frac{|x|^4 + 4(x_1x_2)^2}{6n^3} + O\left( \frac{|x|^5}{n^4} \right) \right\} \]

for \( x = (x_1, x_2) \in \mathbb{Z}^2 \) with \( n + x_1 + x_2 \) even. This formula, however, can be obtained rather directly if one notices that in the frame obtained by rotating the original one by a right angle the two components in the new frame are symmetric simple random walks on \( \mathbb{Z}/\sqrt{2} \) that are independent of each other and use an expansion of transition probability of these walks as given in [8] (Section VII.6, problem 14).

2 Proof of Theorem 1

2.1. Proof of (1.2).

The arguments involved in this subsection partly prepares for the proof of (1.3).

By definition \( \lambda = m(\mu) \) is the inverse function of

\[ \mu = \nabla \phi(\lambda) = E[Xe^{X\lambda}] / E[e^{X\lambda}] = Q\lambda + 1/2 E[(X \cdot \lambda)^2 X] + O(|\lambda|^3), \]

so that

\[ \lambda = m(\mu) = Q^{-1}\mu - 1/2 E[(X \cdot Q^{-1}\mu)^2 Q^{-1}X] + O(|\mu|^3). \] (2.1)

The Taylor expansion of \( \phi \) about the origin up to the third order is given by

\[ \phi(\lambda) = 1/2 Q(\lambda) + 1/6 E[(X \cdot \lambda)^3] + O(|\lambda|^4), \] (2.2)

hence for \( |\mu| \) small enough,

\[ \phi(m(\mu)) = 1/2 Q^{-1}(\mu) - 1/3 E[(Q^{-1}X \cdot \mu)^3] + O(|\mu|^4). \] (2.3)

Here \( Q(\lambda) = \lambda \cdot Q \lambda \), the quadratic form determined by the matrix \( Q \) and similarly \( Q^{-1}(\mu) = \mu \cdot Q^{-1} \mu \).

Now we compute \( H(\mu) \) by using (1.5). From (2.3) and \( \phi(i\theta) = -\frac{1}{2}Q(\theta) + O(|\theta|^3) \) (for \( \theta \) small) it follows that

\[ 1 - e^{-\phi(m(\mu)) + \phi(i\theta)} = \frac{1}{2} [Q^{-1}(\mu) + Q(\theta)] + O(|\mu|^3 + |\theta|^3). \]

Substitution into (1.5) and a simple computation show

\[ \frac{1}{H(\mu)} = \frac{2}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{d\theta}{Q^{-1}(\mu) + Q(\theta) + O(|\mu|^3 + |\theta|^3)} \]

\[ = \frac{-1}{2\pi|Q|^{1/2}} \log Q^{-1}(\mu) + O(1). \]

Noting \( \log Q^{-1}(\mu) = 2 \log |\mu| + O(1) \) we obtain (1.2).

2.2. A local limit theorem.

Let \( q(x) \) denote the probability law of the increment of the walk: \( q(x) = P[X = x] \).

Let \( \mu = \nabla \phi(\lambda) \) with \( \lambda \) in the interior of \( \Xi \) and define

\[ q_\mu(x) = \frac{1}{E[e^{m(\mu)X}]} e^{m(\mu)\cdot x} q(x) \]
The mean number of sites visited

\((m(\mu)\) is defined by (1.1)) so that \(q_\mu\) is a probability on \(\mathbb{Z}^2\) with the mean

\[
\sum xq_\mu(x) = \nabla \phi(m(\mu)) = \mu.
\]

Let \(q^n\) and \(q^n_\mu\) be the \(n\)-fold convolution of \(q\) and \(q_\mu\), respectively. Then

\[
q^n(x) := P[S_n = x] = (E[e^{m(\mu)\cdot X}])^n e^{-m(\mu)\cdot x} q^n_\mu(x).
\]

(2.4)

Let \(Q_\mu\) denote the covariance matrix of the probability \(q_\mu\) and \(Q_\mu^{-1}(x)\) the quadratic form determined by \(Q_\mu^{-1}\).

**Lemma 3.** Let \(K\) be a compact set contained in the interior of \(\Xi\) (as in Theorem 1). Then uniformly for \(y \in \mathbb{Z}^2 - n\mu\) and for \(\mu \in \nabla \phi(K)\), as \(n \to \infty\)

\[
q^n_\mu(n\mu + y) = \frac{\nu 1(q^n(n\mu + y) \neq 0)}{2\pi n \sigma_\mu^2} e^{-Q_\mu^{-1}(y)/2n} \left[ 1 + P_{\mu}^{n,N}(y) \right] + O\left( [y^2 \vee n]^{-N/2} \right).
\]

Here \(N\) may be an arbitrary positive integer, \(\nu\) is the period of the walk \(S_n\), \(1(S)\) is 1 or 0 according as the statement \(\nu\) is true or false, \(\sigma_\mu^2\) denotes the square root of the determinant of \(Q_\mu\) and

\[
P_{\mu}^{n,N}(y) = n^{-1/2} P_1^{\mu}(y/\sqrt{n}) + \cdots + n^{-N/2} P_N^{\mu}(y/\sqrt{n}),
\]

where \(P_\mu^{n}\) is a polynomial of degree at most \(3j\) determined by the moments of \(q_\mu^n\) and odd for odd \(j\).

**Proof.** This lemma may be a standard result. In fact it is reduced to the usual local central limit theorem as follows. Let \(\psi_\mu(\theta)\) be the characteristic function of \(q_\mu\) and put

\[
\psi_\mu(\theta) := \sum x q_\mu(x) e^{i\theta \cdot x} = \hat{\psi}_\mu(\theta) e^{i\mu \cdot \theta}.
\]

Hence

\[
q^n_\mu(n\mu + y) = \frac{1}{(2\pi)^2} \int_T \left[ \hat{\psi}_\mu(\theta) \right]^n e^{-i(y + \theta \cdot n\mu) \cdot \theta} d\theta
\]

\[
= \frac{1}{(2\pi)^2} \int_T \left[ \hat{\psi}_\mu(\theta) \right]^n e^{-i\theta \cdot y} d\theta,
\]

(2.5)

where \(T = [-\pi, \pi] \times [-\pi, \pi]\). Since \(\nabla \hat{\psi}_\mu(0) = 0\), the Hessian matrix of \(\hat{\psi}_\mu\) at zero equals \(Q_\mu\) and \(\sum p_n(x) |x|^{2N} < \infty\) for all \(N > 0\), the usual procedure to derive the local limit theorem (see [9]; also Appendix (A) for the case \(\nu > 1\) if necessary) shows that the right-most member equals that of the formula of the lemma.

Define \(\Lambda \subset \mathbb{Z}^2\) by

\[
\Lambda = \{ x \in \mathbb{Z}^2 : q^n_\mu(x) \neq 0 \text{ for some } n \}.
\]

(2.6)

Plainly \(\Lambda\) is a subgroup of \(\mathbb{Z}^2\). Take an \(x \in \mathbb{Z}^2\) with \(q(x) > 0\) and put \(\Lambda_k = \Lambda + k\xi\), the shift of \(\Lambda\) by \(k\xi\). \(\Lambda_k\) does not depend on the choice of \(k\) and is periodic in \(k\) of period \(\nu\). It holds that \(P[S_n \in \Lambda_k] > 0\) only if \(n = k \mod(\nu)\). In the formula of Lemma 3 the trivial factor \(1(q^n(n\mu + y) \neq 0)\) may be replaced by \(1(n\mu + y \in \Lambda_k)\); also, for each \(k \in \mathbb{Z}\), \(q^n_\mu(n\mu + y)\) may be replaced by \(q^n_{\mu+k}(n\mu + y)\), hence by \(q^n_{\mu+k}(n\mu + y)\). Thus we can reformulate Lemma 3 as in the following

**Corollary 4.** Let \(K\) be a compact set contained in the interior of \(\Xi\). Then for each \(k \in \mathbb{Z}\), uniformly for \(y \in \mathbb{Z}^2 - n\mu\) and for \(\mu \in \nabla \phi(K)\), as \(n \to \infty\)

\[
q^n_{\mu+k}(n\mu + y) = \frac{\nu 1(n\mu + y \in \Lambda_{n+k})}{2\pi n \sigma_\mu^2} e^{-Q_\mu^{-1}(y)/2n} \left[ 1 + P_{\mu}^{n,N}(y) \right] + O\left( [y^2 \vee n]^{-N/2} \right),
\]

with the same notation as in Lemma 3.
The mean number of sites visited

Proof of Proposition 2. In Lemma 3 we take \( \mu = x/n \). It follows that with \( \lambda = m(\mu) \)

\[
Q_{\mu} = \nabla \log \phi(\lambda) + [\nabla \phi(\lambda)/\phi(\lambda)]^2 = Q + O(|\mu|),
\]
so that \( \sigma^2_{\mu} = \sigma^2 + O(|\mu|) \). In view of (2.4) and Lemma 3 we have only to compute asymptotic form of

\[
E[e^{m(\mu)X}] e^{-m(\mu)x} = \exp\{m(\mu) - m(\mu) \cdot \mu\}.
\]

By (2.1) and (2.3)

\[
\phi(m(\mu)) - m(\mu) \cdot \mu = -\frac{1}{2} Q^{-1}(\mu) + \frac{1}{6} E[(Q^{-1}X \cdot \mu)^3] + \kappa_4(\mu) + O(|\mu|^5)
\]

for \( |\mu| \) small enough, where \( \kappa_4(\mu) \) is a polynomial of degree 4.

Assume that all the third moments of \( X \) vanish. Then, in place of (2.1) and (2.2) we have

\[
\lambda = m(\mu) = Q^{-1} \mu + b(\mu),
\]

with \( b(\mu) = O(|\mu|^3) \) and

\[
\phi(\lambda) = \frac{1}{2} Q(\lambda) - \frac{1}{8} [Q(\lambda)]^2 + \frac{1}{24} E[(X \cdot \lambda)^4] + O(|\lambda|^5),
\]

respectively. Substituting these formulae into \( m(\mu) \cdot \mu - \phi(m(\mu)) \) we observe that the term involving \( b(\mu) \) disappears from the fourth order term by cancellation and hence that

\[
\phi(m(\mu)) - m(\mu) \cdot \mu = -\frac{1}{2} Q^{-1}(\mu) - \frac{1}{8} [Q(\lambda)]^2 + \frac{1}{24} E[(Q^{-1}X \cdot \mu)^3] + O(|\mu|^5),
\]

in which we find the explicit form of \( \kappa_4(\mu) \) as presented in the proposition. \( \square \)

2.3. Proof of (1.3).

The proof is based on the identity

\[
E[Z; S_n = x] = n q^n(x) - \sum_{k=1}^{n-1} f_0(k) q^{n-k}(x)(n-k)
\]

(cf. [12], Lemma 1.1) as well as Corollary 4. Let \( q^n(x) \neq 0 \). Remembering \( E[e^{m(\mu)X}] = e^{\phi(m(\mu))} \) we obtain from (2.4) that

\[
\frac{q^{n-k}(x)(n-k)}{q^n(x)n} = e^{-k \phi(m(\mu))} \frac{q^{n-k}(x)(n-k)}{q^0_n(x)n}.
\]

On writing \( \mu := x/n \) and \( x = (n-k) \mu + k \mu \), Corollary 4 gives

\[
q^{n-k}_n(x)(n-k) = \frac{\nu 1(x \in A_{n-k}) e^{-Q_{\mu}^{-1}(k\mu)/2(n-k)} [1 + P_{\mu}^{n-k,N}(k\mu)]}{2\pi \sigma^2_{\mu}} + O(|k\mu|^2 \vee (n-k)^{-N})
\]

and

\[
q^0_n(x)n = \frac{\nu}{2\pi \sigma^2_{\mu}} [1 + O(1/n)].
\]

Let \( 1/\sqrt{n} \leq |\mu| \) and \( \mu \in \nabla \phi(K) \). Noting that \( \sigma^2_{\mu} \) is then bounded away from zero for \( \mu \in \nabla \phi(K) \) we see

\[
\frac{q^{n-k}(x)(n-k)}{q^n(x)n} = 1(x \in A_{n-k}) e^{-k \phi(m(\mu))} e^{-Q_{\mu}^{-1}(k\mu)/2(n-k)} [1 + O(1/\sqrt{n})]
\]

\[
+ O\left(e^{-k \phi(m(\mu))} n^{-N}\right).
\]


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The mean number of sites visited

Since $\sum_{k>n^{1/3}} f_0(k) = O(1/\log n)$, it follows that

$$E[Z_n; S_n = x] = nq^n(x) \left[ \sum_{k=1}^{n^{1/3}} f_0(k) \left( 1 - 1(\mathbf{x} \in \Lambda_{n-k}) e^{-k\phi(c(x/n))} \right) + O\left( \frac{1}{\log n} \right) \right].$$

Under the condition $q^n(x) \neq 0$, it follows from $f_0(k) \neq 0$ that $x \in \Lambda_{n-k}$. Hence

$$E[Z_n | S_n = x] = n \sum_{k=1}^{\infty} f_0(k) \left( 1 - e^{-k\phi(c(x/n))} \right) + O\left( \frac{n}{\log n} \right). \tag{2.11}$$

We still need to obtain the error bound $O(n/|\log \mu|^2)$ instead of $O(n/\log n)$. To this end, on applying the asymptotic formula

$$f_0(k) = \frac{2\pi |Q|^{1/2}}{k(\log k)^2} + O\left( \frac{1}{k(\log k)^3} \right)$$

(cf. [13]) we see, on the one hand, that for $0 < \phi < 1/2$

$$\sum_{k>\delta/\phi} f_0(k) e^{-k\phi} = O\left( \frac{1}{(\log \phi)^2} \right), \tag{2.12}$$

where $\delta$ is an arbitrarily fixed positive constant, and by using (2.3), on the other hand, we see

$$\phi(m(\mu)) > c|\mu|^2 \geq c/n$$

(the second inequality is nothing but our present supposition that $|x| \geq \sqrt{n}$). As in a similar way to the derivation of (2.11) we deduce from (2.9) with the help of (2.12) as well as of (2.10) that

$$\frac{E[Z_n; S_n = x]}{nq^n(x)} = \sum_{k=1}^{\infty} f_0(k) \left( 1 - e^{-k\phi(c(x/n))} \right) + O\left( \frac{1}{(\log |\mu|)^2} \right),$$

if it is true that as $\mu \to 0$

$$\sum_{k<\epsilon/2\phi(m(\mu))} f_0(k) e^{-k\phi(m(\mu))} \left( 1 - e^{-Q^{-1}(k\mu)/2(n-k)} \right) = O(1/(\log |\mu|)^2). \tag{2.13}$$

Since $c/2\phi(m(\mu)) \leq n/2$, the sum on the left-hand side of (2.13) is at most a constant multiple of

$$\sum_{k<\epsilon/2\phi(m(\mu))} f_0(k) \frac{Q^{-1}(k\mu)}{n} \leq \frac{Q^{-1}(\mu)}{n} \sum_{k<\epsilon/2\phi(m(\mu))} f_0(k) k^2 \leq \frac{c|\mu|^2}{n} \left( \frac{k^2}{(\log k)^2} \right)_{k=\epsilon/2\phi(m(\mu))} = O\left( \frac{1}{(n(\log |\mu|)^2} \right).$$

verifying (2.13) (with a better bound).

Thus we have proved (1.3) and hence Theorem 1.

3 Appendix

(A) In the case when the period $\nu$ is larger than 1 the evaluation of the integral in (2.5) is reduced to that for the case $\nu = 1$ by consideration of a property of its integrand that reflects the periodicity. By an elementary algebra one can find a point $\eta \in \mathbb{R}^2$ that satisfies that for $j = 0, 1, \ldots, \nu - 1$,

$$\eta \cdot \mathbf{x} - j\nu^{-1} \in \mathbb{Z} \quad \text{if} \quad \mathbf{x} \in \Lambda_j$$
where \( \mu \) is defined shortly after (2.6). From this relation it follows that
\[
\psi(\theta + 2\pi k \eta) = \psi(\theta)e^{2i\pi k / \nu} \quad (k = 0, \ldots, \nu - 1).
\]

Now consider the expression \( q^\alpha(x) = (2\pi)^{-2} \int_q[\psi(\theta)]n e^{-ix \theta} d\theta \). Observe that if \( x \in \Lambda_j \),
\[
[\psi(\theta + 2\pi k \eta)]n e^{-ix \theta} = [\psi(\theta)]n e^{-ix \theta} e^{2\pi(n-j)k / \nu} \quad (k = 0, \ldots, \nu - 1)
\]
and the right-hand sides equal \( [\psi(\theta)]n e^{-ix \theta} \) for all \( k \) if \( n-j \) equals zero in mod \( \nu \), while their sum over \( k \) vanishes otherwise. Choosing \( \varepsilon > 0 \) small enough, we may replace \( 2\pi \eta \) by a unique \( \eta_k \in [-1, 1] \) such that \( \eta_k - \eta \in \mathbb{Z}^2 \) and apply the usual method for evaluation of Fourier integral.

(B) Put \( \psi(\lambda) = E[e^{i\lambda X}] \), so that \( \mu = \nabla \phi(\lambda) = E[X e^{i\lambda X}] / \psi(\lambda) \). At \( \lambda = m(\mu) \) we have
\[
\nabla^2 \phi(\lambda) = E[X^2 e^{i\lambda X}] / \psi(\lambda) - \mu^2 = Q_{\mu},
\]
where \( \mu^2 \) is understood to be \( 2 \times 2 \) matrix: \( \mu^2 = (\mu_i \mu_j)_{1 \leq i, j \leq 2} \), and similarly for \( X^2 \) and \( \nabla^2 \). Since \( \text{id} = \nabla m(\mu) \frac{\partial}{\partial \mu} = \nabla m(\mu) \nabla^2 \phi(m(\mu)) = \nabla m(\mu) Q_{\mu} \), it holds that
\[
\nabla m(\mu) = Q_{\mu}^{-1}.
\]

Therefore, from the defining formula of \( H \) we have
\[
\nabla H(\mu) = C(\mu) \nabla (\phi \circ m)(\mu) = C(\mu) Q_{\mu}^{-1} \mu,
\]
where
\[
C(\mu) = \sum_{k=1}^{\infty} k f_0(k) e^{-k \phi(m(\mu))}.
\]

Let \( E^\mu \) designate the expectation w.r.t. \( q_\mu \), i.e.,
\[
E^\mu[\cdot] = \left[ E[\cdot e^{i\lambda X}] / \psi(\lambda) \right]_{\lambda = m(\mu)}.
\]

Then
\[
Q_{\mu}^{-1} \mu = \frac{1}{\det Q_{\mu}} \begin{bmatrix}
E^\mu[X_2^2] - \mu_2^2 & \mu_1 \mu_2 - E^\mu[X_1 X_2] \\
\mu_1 \mu_2 - E^\mu[X_1 X_2] & E^\mu[X_1^2] - \mu_1^2
\end{bmatrix}
\]
\[
= \frac{1}{\det Q_{\mu}} \begin{bmatrix}
\mu_1 E^\mu[X_2^2] - \mu_2 E^\mu[X_1 X_2] \\
\mu_2 E^\mu[X_1^2] - \mu_1 E^\mu[X_1 X_2]
\end{bmatrix}.
\]

Hence
\[
\nabla H(\mu) \cdot \mu = \frac{1}{\det Q_{\mu}} C(\mu) \left[ \mu_1 E^\mu[X_2^2] - \mu_2 E^\mu[X_1 X_2] + (\mu_1^2 - \mu_2^2) E^\mu[X_1 X_2] \right].
\]

For the simple random walk in Example 1, \det \( Q_{\mu} = (\cosh \alpha + \cosh \beta)^{-2}, E^\mu[X_1 X_2] = 0 \) and
\[
E^\mu[X_2^2 - X_1^2] = \frac{\cosh \beta - \cosh \alpha}{2\psi(m(\mu))} = \frac{\sinh^2 \beta - \sinh^2 \alpha}{(\cosh \alpha + \cosh \beta)^2} = \mu_2^2 - \mu_1^2 \quad (\lambda = m(\mu)),
\]
showing the last formula of Example 1 with \( C_0(\mu) = C(\mu)(\cosh \alpha + \cosh \beta)^2 \).
The mean number of sites visited

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