Gaussian integrability of distance function under the Lyapunov condition

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Abstract

In this note we give a direct proof of the Gaussian integrability of distance function as
\[ \mu e^{\delta d^2(x,x_0)} < \infty \]
for some \( \delta > 0 \) provided the Lyapunov condition holds for symmetric diffusion operators, which answers a question in Cattiaux-Guillin-Wu [6, Page 295]. The similar argument still works for diffusions processes with unbounded diffusion coefficients and for jump processes such as birth-death chains. An analogous discussion is also made under the Gozlan’s condition arising from [9, Proposition 3.5].

Keywords: Gaussian integrability; Lyapunov condition; diffusion process; jump process.

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1 Introduction

In this note, we will investigate how to directly derive the Gaussian integrability from two kinds of criteria for the Talagrand’s inequality \( W_2 H \) (or \( T_2 \) in short), say the Lyapunov condition and Gozlan’s condition presented in a symmetric diffusion Markov setting. Referring to Bakry-Gentil-Ledoux [2], we denote by \( E \) a complete connected Riemannian manifold of finite dimension, \( d \) the geodesic distance, \( dx \) the volume measure, \( \mu(dx) = e^{-V(x)}dx \) a probability measure with \( V \in C^2(E) \), \( L = \Delta - \nabla V \cdot \nabla \) the \( \mu \)-symmetric diffusion operator, \( \Gamma(f,g) = \nabla f \cdot \nabla g \) the carré du champ operator, and \( \mathcal{E} \) the associated Dirichlet form, which satisfy the formula for integration by parts

\[
\int \nabla f \cdot \nabla g d\mu = -\int fLg d\mu, \quad f \in D(\mathcal{E}), g \in D(L).
\]

First of all, say \( W \geq 1 \) is a Lyapunov function if there exist two constants \( b \geq 0 \) and \( c > 0 \) such that for some \( x_0 \in E \) and any \( x \in E \)

\[
LW \leq (-c d^2(x,x_0) + b)W. \tag{1.1}
\]

More generally, to avoid assuming the integrability and second-order differentiability of \( W \), it is convenient to introduce a locally Lipschitz function \( U > 0 \) such that in the sense of distribution

\[
LU + |\nabla U|^2 \leq -c d^2(x,x_0) + b, \tag{1.2}
\]

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which means that for any nonnegative \( h \in C_0^\infty(E) \) holds

\[
\int \left( LU + |\nabla U|^2 \right) h d\mu := \int ULh + |\nabla U|^2 h d\mu \leq \int \left( -cd^2(x, x_0) + b \right) h d\mu.
\]

When \( W \in C^2(E) \), (1.1) and (1.2) are equivalent by taking \( U = \log W \). And it is not hard to see that (1.1) implies a weaker version for some \( c', b' \) and \( \overline{R} \)

\[
\text{LW} \leq -c'W + b'1_{B(0, \overline{R})}.
\]  

The Lyapunov condition plays a powerful role in studying functional inequalities or estimating convergence rate of Markov processes, which even works as a substitute of curvature-dimension condition sometimes. Cattiaux-Guillin [4] gave a comprehensive review on this topic, and here we would like to take partial literature into account. A simple proof of the Poincaré inequality through (1.3) can be found in Bakry-Barthe-Cattiaux-Guillin [1]. The \( L^1 \) transport-information inequality \( W_1I \) was discussed further under (1.1) by Guillin-Léonard-Wu-Yao [12]. Then Cattiaux-Guillin-Wu [6] showed the Talagrand’s inequality and logarithmic Sobolev inequality (LSI for short) provided (1.2), which was also applied to weighted LSIs for heavy tailed distributions by [7]. Most recently, Guillin-Joulia [10] obtained non-Gaussian concentration estimates by means of functional inequalities with some kind of Lyapunov condition yet.

According to [6, Lemma 3.5], it was proved that if (1.2) holds, there exist some \( \delta > 0 \) and \( x_0 \in E \) such that

\[
\int e^{\delta d^2(x, x_0)} d\mu(x) < \infty
\]

which is necessary to derive \( W_2H \). Their proof starts from (1.2) and the spectral gap to show \( W_1I \) due to [12]. It then follows a \( L^1 \) transport-entropy inequality \( W_1H \) by Guillin-Léonard-Wang-Wu [11], which is equivalent to (1.4) by Djellout-Guillin-Wu [8]. The strategy relies on a series of works on transport inequalities, thereupon the authors of [6] feel interested in finding a simple or direct proof of (1.4), see [6, Page 295].

Indeed, there exists an elementary proof, and we actually obtain

**Proposition 1.1.** If (1.2) holds, then \( \mu e^{\delta d^2(x, x_0)} < \infty \) for any \( \delta < \sqrt{\gamma} \).

**Remark 1.2.** The upper bound for \( \delta \) is sharp. For instance, let \( d\mu = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}|x|^2} \) and \( L = \frac{d^2}{dx^2} - x \frac{d}{dx} \) associated to one-dimensional Ornstein-Uhlenbeck process, then \( W = e^{\frac{1}{4}|x|^2} \) satisfies \( \text{LW} \leq (-\frac{1}{4}|x|^2 + \frac{1}{2})W \), which exactly gives \( \delta < \sqrt{\gamma} = \frac{1}{2} \).

**Remark 1.3.** A weak version \( \text{LW} \leq (-cd^p(x, x_0) + b)W \) with \( p < 2 \) is not enough to derive the Gaussian integrability, since \( W = \exp \left( \frac{1}{2} (1 + |x|^2)^{q/2} \right) \) with \( 2(q - 1) = p \) fulfills (1.2) with respect to \( d\mu = \frac{1}{2} e^{-(1+|x|^2)\frac{q}{2}} \) \( dx \), where \( Z \) is a normalized factor.

The same argument can be extended to diffusion processes with unbounded diffusion coefficients. Define an infinitesimal generator in \( \mathbb{R}^m \)

\[
L_a = \frac{1}{2} \sum_{i,j=1}^m a^{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i=1}^m b^i(x) \frac{\partial}{\partial x_i},
\]

where \( A = (a^{ij})_{i,j=1}^m \) is symmetric positive definite and \( b' = \frac{1}{2} \left( \sum_{j=1}^m \frac{\partial a_{ij}}{\partial x_j} - a_{ij} \frac{\partial V}{\partial x_j} \right) \) so that \( L_a \) admits an invariant probability measure \( d\mu(x) = e^{-V} dx \). Then define the Carré du champ operator by means of

\[
\Gamma_a(f, g) = \frac{1}{2} \left[ L_a(fg) - fL_ag - gL_af \right] = \frac{1}{2} \langle \nabla f, A \nabla g \rangle,
\]

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which satisfies the integration by parts formula for \( f, g \in C_c^\infty(\mathbb{R}^m) \)

\[
- \int f L_\nu g \, d\mu = \int \Gamma_\nu(f, g) \, d\mu =: \mathcal{E}_\nu(f, g).
\]

Thanks to the Lyapunov type criterion by Stroock-Varadhan [13, Theorem 10.2.1], it can be quickly derived that \( L_\nu \) corresponds to a non-explosive diffusion process provided that (1.1) holds by substituting \( L \) to \( L_\nu \)

\[
L_\nu W \leq (-cd^2(x, x_0) + b)W
\]

with \( \lim_{|x| \to \infty} W = \infty. \)

However, if \( a^{ij} \) is unbounded, (1.5) is not enough to get the Gaussian integrability for \( \mu \). Consider one-dimensional case, when \( a^{ij} = a(x) = o(|x|^4) \), we take \( V = \frac{x^2}{4c^2} \) and \( W = e^{\delta V} \) for small \( \delta > 0 \) so that (1.5) holds but \( V \) has a growth rate slower than quadratic. On the other hand, when \( a(x) = O(|x|^4) \) grows even faster, (1.5) is useless to yield a Poincaré type inequality so that we have no effective calculus on the integrability of \( e^{\delta d^2(x, x_0)}. \) For this reason, a stronger condition is necessary.

**Proposition 1.4.** Let \( \lambda_{\text{max}} \) be the maximal eigenvalue of \( A \) satisfying \( \mu \lambda_{\text{max}} < \infty. \) Suppose there exists a Lyapunov function \( W \geq 1 \) with two constants \( b \geq 0 \) and \( c > 0 \) such that for some \( x_0 \in \mathbb{R}^m \) and any \( x \in \mathbb{R}^m \)

\[
L_\nu W \leq (-cd^2(x, x_0) + b)\lambda_{\text{max}}W.
\]

Then \( \mu \left( e^{\delta d^2(x, x_0)} \lambda_{\text{max}} \right) < \infty \) for any \( \delta < \sqrt{c}. \)

**Remark 1.5.** (1.6) is natural, for instance, if there exist \( V \) and \( W \) satisfying (1.1) over \( \mathbb{R} \), then (1.6) follows automatically provided that \( \lim_{|x| \to \infty} \frac{W}{W|x|^2} = 0. \) Moreover, there is no need to assume \( \lambda_{\text{max}} \geq \lambda > 0 \) uniformly on \( \mathbb{R}^m. \)

Another possible extension is about jump processes (see Bass [3]). To clarify the effect from jumps part, we simply consider the infinitesimal generator of the form

\[
L_\nu = \int_{\mathbb{R}^n - \{0\}} [f(x + y) - f(x) - \nabla f \cdot y 1_{0 < |y| < 1}(y)] \nu(x, dy),
\]

Where \( \nu \) satisfies \( \int_{\mathbb{R}^n - \{0\}} \min\{1, |y|^2\} \nu(x, dy) < \infty. \) Suppose that \( L_\nu \) admits an invariant probability measure \( \mu, \) and the Carrédu champ operator

\[
\Gamma_\nu(f, g) = \frac{1}{2} \left[ L_\nu(fg) - fL_\nu g - gL_\nu f \right] = \frac{1}{2} \int_{\mathbb{R}^n - \{0\}} [f(x + y) - f(x)] [g(x + y) - g(x)] \nu(x, dy)
\]

fulfills the integration by parts formula for \( f, g \in C_c^\infty(\mathbb{R}^m). \)

\[
- \int f L_\nu g \, d\mu = \int \Gamma_\nu(f, g) \, d\mu =: \mathcal{E}_\nu(f, g).
\]

Then define an intrinsic (pseudo)metrical according to Sturm [14, Definition 6.5]

\[
\rho(x, y) := \sup\{f(x) - f(y) : \Gamma_\nu(f, f) \leq 1\},
\]

which gives \( \Gamma_\nu(\rho(x, x_0), \rho(x, x_0)) \leq 1 \) if \( \rho(x, x_0) \in \mathcal{D}(\mathcal{E}_\nu). \) For convenience, we also require that \( \lim_{|x| \to \infty} \rho(x, x_0) = \infty. \)

The setting includes discrete Markov chains. For example, consider a birth-death process on \( \mathbb{N} \) with strictly positive birth rates \( b_i \) and death rates \( d_i \) except \( d_0 = 0. \)
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Let \( r_0 = 1 \) and \( r_i = \frac{b_i h_{i-1} - b_i}{a_i} \) for \( i \geq 1 \), we can take \( \nu(i,y) = b_i \delta_1(y) + d_i \delta_{-1}(y) \) and \( \mu(i) = \frac{a_i}{(a_i + r_i)^{1+\epsilon}} \) provided the series converges, and then \( \mathcal{E}_\nu \) has an alternative expression \( \mathcal{E}_\nu(f,g) = \sum_{i=0}^{\infty} [f(i+1) - f(i)] [g(i+1) - g(i)] b_i \mu_i \), which determines the intrinsic metric \( \rho(i,j) = b_i^{-\frac{1}{2}} + b_{i+1}^{-\frac{1}{2}} + \cdots b_{j-1}^{-\frac{1}{2}} \) for \( i \leq j \).

**Proposition 1.6.** Suppose there exist some \( x_0 \in \mathbb{R}^m \) and a constant \( K > 0 \) such that for all \( x \in \mathbb{R}^m \) and all \( y \in \text{Supp} \nu \)

\[
|\rho^2(x + y, x_0) - \rho^2(x, x_0)| \leq K. \tag{1.7}
\]

Suppose also there exists a Lyapunov function \( W \geq 1 \) with two constants \( b \geq 0 \) and \( c > 0 \) such that for any \( x \in \mathbb{R}^m \)

\[
L_\nu W \leq (-c \rho^2(x, x_0) + b) W. \tag{1.8}
\]

Then \( \mu e^{\delta^2(x,x_0)} < \infty \) for \( \delta < C \min \{ \sqrt{c}, K^{-1} \} \) with some multiple \( C \in (0,1] \).

**Remark 1.7.** For a birth-death process referring to Cattiaux-Guillin-Wang-Wu [5], let \( b_i = d_i = \alpha i \log^a(i+1) \) with \( a \geq 2 \) and \( \alpha \in \mathbb{R} \) except \( b_0 = 1 \), let \( W = 1 + i^\gamma \) with \( 0 < \gamma < 1 \), then \( \mu(i) \propto b_i^{-1} \) and \( L_\nu W \leq c^a i^{-2 \log^a(i+1) W} \). Take \( a = 2, \alpha = 1, \gamma = \frac{1}{2} \) explicitly, it follows \( \rho(i,0) \propto \log^\frac{2}{3}(i+1) \) satisfying (1.7-1.8) and then \( \mu e^{\delta^2} < \infty \) for any \( \delta < 1 \). The combination of (1.7) and (1.8) is necessary. When \( a = 2, \alpha < 1, \gamma = \frac{1}{2} \), (1.7) still holds, but (1.8) fails and so does the Gaussian integrability. On the other hand, let \( b_i = (i+1)^\frac{2}{3} \) and \( d_i = ib_i \), then \( \mu(i) \propto (ib_i)^{-1}, \rho(i,0) \propto i^{\frac{2}{3}} \), and (1.8) holds for \( W = i^2 \), but (1.7) fails and so does the Gaussian integrability again.

We further investigate another criterion for transport inequalities. According to Gozlan [9, Proposition 3.5], let \( \mu \) be a probability on \( \mathbb{R}^m \), suppose there exists \( \omega \in C^3(\mathbb{R}) \) with \( \omega' > 0 \), \( \frac{\omega^{(3)}}{\omega'^3} \leq M \) for some constant \( M \), and

\[
\liminf_{|x| \to \infty} \frac{1}{u^2} \sum_{i=1}^{m} \left[ \frac{1}{10} \left( \frac{\partial V}{\partial x_i} \right)^2 \left( \frac{x}{u} \right) - \frac{\partial^2 V}{\partial x_i^2} \left( \frac{x}{u} \right) \right] \frac{1}{\omega'(x_i)^2} \geq mM \tag{1.9}
\]

for some constant \( u > 0 \), then a transport-entropy inequality holds with the cost function

\[
d_{\omega}(x, y) = \left( \sum_{i=1}^{m} |\omega(x_i) - \omega(y_i)|^2 \right)^{\frac{1}{2}}. \]

An interesting case is to set

\[
\omega(t) = \int_0^t \sqrt{1 + s^2} ds = \frac{t}{2} \sqrt{1 + t^2} + \frac{1}{2} \log \left( t + \sqrt{1 + t^2} \right)
\]

satisfying \( \omega'(0) = 1 \) and \( \frac{\omega^{(3)}}{\omega'^3}(t) = (1 + t^2)^{-3} \leq 1 \), which corresponds to \( W_2 H \).

In [6], it was pointed out that (1.9) is not comparable to the Lyapunov condition (1.2) in general. Using the similar argument, we still have

**Proposition 1.8.** If the Gozlan’s type condition holds, i.e.

\[
\liminf_{|x| \to \infty} \sum_{i=1}^{m} \left[ \frac{23}{27} \left( \frac{\partial V}{\partial x_i} \right)^2 \left( x \right) - \frac{\partial^2 V}{\partial x_i^2} \left( x \right) \right] \frac{1}{\left( 1 + x_i^2 \right)^{\frac{1}{2}}} \geq m, \tag{1.10}
\]

then \( \mu e^{\delta|x|^2} < \infty \) for any \( \delta < \frac{2(\sqrt{m} - \sqrt{m-1})}{3 \sqrt{3m}} \).

**Remark 1.9.** To yield the Gaussian integrability, or equivalently \( W_1 H \), the original constant \( \frac{1}{12} \) in (1.9) can be increased to arbitrary \( \alpha < 1 - \frac{1}{2} \frac{m-1}{m} \). So it is convenient to take \( \alpha = \frac{27}{5} \). Except \( m = 1 \), it is unlikely to allow \( \alpha \) approaching 1, according to the estimates in Lemma 3.1 below.

The next two sections will supply the proofs of all propositions respectively.

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2 Proofs of Proposition 1.1, 1.4 and 1.6

Under the Lyapunov condition (1.2), [6, Lemma 3.4] asserts
\[ \int h^2(x) d^2(x, x_0) d\mu(x) \leq \frac{1}{c} \int |\nabla h|^2 d\mu + \frac{b}{c} \int h^2 d\mu, \quad \forall h \in \mathcal{D}(\mathcal{E}), \quad (2.1) \]
basically via the same technique as in [1, Page 64].

**Proof of Proposition 1.1.** Let \( \beta_0 = \int d^{2n}(x, x_0) d\mu \), which satisfies a recursion by using (2.1) that
\[ \beta_n = \int d^{2(n-1)}(x, x_0) d^2(x, x_0) d\mu \leq \frac{1}{c} \int |\nabla d^{n-1}(x, x_0)|^2 d\mu + \frac{b}{c} \beta_{n-1} = \frac{(n-1)^2}{c} \beta_{n-2} + \frac{b}{c} \beta_{n-1}. \] (2.2)

Since \( \beta_0 = 1 \) and \( \beta_1 \leq \frac{b}{c} \), we get the integrability of all \( d^{2n}(x, x_0) \).

Combining the Hölder inequality with (2.2) gives
\[ \beta_n = \int d^{n+1}(x, x_0) d^{n-1}(x, x_0) d\mu \leq \beta_{n+1}^\frac{1}{2} \beta_{n-1}^\frac{1}{2} \leq \left( \frac{n^2}{c} \beta_{n-1} + \frac{b}{c} \beta_n \right)^\frac{1}{2} \beta_{n-1}^\frac{1}{2}, \]
which implies
\[ \beta_n \leq \frac{b}{c} + \frac{n^2}{2c} \beta_{n-1} \leq (\frac{b}{c} + \frac{n}{\sqrt{c}}) \beta_{n-1}. \]

Taking any \( \gamma > \frac{1}{\sqrt{c}} \) gives \( \frac{b}{c} + \frac{n}{\sqrt{c}} \leq \gamma n \) for big \( n \), which yields some \( C > 0 \) such that
\[ \beta_n \leq C \gamma^n n!, \quad \forall n \geq 1. \]

Hence, for any \( \delta < \gamma^{-1} < \sqrt{c} \), we have by the Fatou’s lemma
\[ \int e^{\delta d^2(x, x_0)} d\mu = \int \lim_{k \to \infty} \sum_{n=0}^{k} (\delta d^n(x, x_0))^n / n! \ d\mu = \lim_{k \to \infty} \sum_{n=0}^{k} \delta^n \beta_n / n! \leq \frac{C}{1 - \delta \gamma}. \] (2.3)

The proof is completed. \( \square \)

The next proof is almost the same.

**Proof of Proposition 1.4.** Using the Lyapunov condition (1.6) with the technique from [1, Page 64] gives a similar inequality for \( h \in \mathcal{D}(\mathcal{E}_u) \) as (2.1) that
\[ \int h^2(x) d^2(x, x_0) \lambda_{\max} d\mu \leq \frac{1}{c} \int h^2 \cdot - \frac{L_a W}{W} d\mu + \frac{b}{c} \int h^2 \lambda_{\max} d\mu \]
\[ = \frac{1}{c} \int \Gamma_a(\frac{h^2}{W}, W) d\mu + \frac{b}{c} \int h^2 \lambda_{\max} d\mu \]
\[ = \frac{1}{c} \int \Gamma_a(h, h) - W^2 \Gamma_a(h, h) d\mu + \frac{b}{c} \int h^2 \lambda_{\max} d\mu \]
\[ \leq \frac{1}{c} \int |\nabla h|^2 \lambda_{\max} d\mu + \frac{b}{c} \int h^2 \lambda_{\max} d\mu. \]
Let \( \beta_n = \int d^{2n}(x, x_0) \lambda_{\max} d\mu \), which satisfies

\[
\beta_n = \int d^{2(n-1)}(x, x_0) d^2(x, x_0) \lambda_{\max} d\mu \\
\leq \frac{1}{c} \int |\nabla d^{n-1}(x, x_0)|^2 \lambda_{\max} d\mu + \frac{b}{c} \beta_{n-1} + \frac{b}{c} \beta_{n-1}.
\]

Following rest steps in the previous proof, we get the Gaussian integrability.

For jump processes, we use a little different method.

**Proof of Proposition 1.6.** The strategy contains three steps.

**Step 1.** Denote \( \rho_t(x) = \sqrt{\rho^2(x, x_0) + t} \) with a parameter \( t > 0 \). Using the technique in [1, Page 64] again, we have by Condition (1.8) that for \( h \in \mathcal{D}(\mathcal{E}_\nu) \)

\[
\int h^2 \rho_t^2 d\mu \leq \frac{1}{c} \int h^2 \cdot \frac{-L_0 W}{W} d\mu + \left( \frac{b}{c} + t \right) \int h^2 d\mu \\
= \frac{1}{c} \int \Gamma_\nu \left( \frac{h^2}{W}, W \right) d\mu + \frac{b + ct}{c} \int h^2 d\mu \\
= \frac{1}{c} \cdot \frac{1}{2} \int_{\mathbb{R}^n} - \left\{ h(x+y) - h(0) + h(x) \right\} \frac{W(x) - W(x+y)}{W(x)^{1/2}} \left( \frac{W(x+y)}{W(x)} \right)^{1/2} \\
+ |h(x+y) - h(0)|^2 \nu(x, dy) \mu(dx) + \frac{b + ct}{c} \int h^2 d\mu \\
\leq \frac{1}{c} \int \Gamma_\nu (h, h, h) d\mu + \frac{b + ct}{c} \int h^2 d\mu.
\]

**Step 2.** Basically, our aim is to estimate \( \int_\Omega e^{\delta \rho(x, x_0)^2} d\mu(x) \) for any bounded domain \( \Omega \), while the integration by parts requires to regularize the characteristic function \( \chi_\Omega \). It is usually a routine but with a few tricks in this case.

Define a family of \( \phi_r \in C^1(\mathbb{R}^n) \) with any \( r > 0 \) and some constant \( N > 0 \) as

\[
\phi_r(s) = \begin{cases} 
1, & s \leq r; \\
2(\frac{s-r}{N})^3 - 3(\frac{s-r}{N})^2 + 1, & r < s < r + N; \\
0, & s \geq r + N,
\end{cases}
\]

which satisfies \( 0 \leq \phi_r \leq 1 \) and \( |\phi'_r| \leq \frac{3}{N} 1_{r<s<r+N} \).

Let \( f = e^{\frac{s^2}{2} \phi_r^2} \) and \( f_r = \phi_r(\rho_t^2) f \). Let \( h_r = \frac{h}{\rho_t} \), we have by Step 1

\[
\int f_r^2 d\mu = \int h_r^2 \rho_t^2 d\mu \leq \frac{1}{c} \int \Gamma_\nu (h_r, h_r) d\mu + \frac{b + ct}{c} \int h_r^2 d\mu.
\]  

(2.4)

For convenience, rewrite \( h_r = \phi_r(\rho_t^2) \psi(\rho_t) \) by putting \( \psi(s) := e^{\frac{s^2}{2} \phi_r^2} \).

Take \( t = 2 \delta^{-1} \) so that \( \psi \) is increasing on \( [\delta, \infty) \). Using the mean value theorem respectively to \( \psi \) and \( \phi_r \) yields that for any \( x \in \mathbb{R}^m \) and \( y \in \text{Supp} \nu \), there exist \( \xi \) and \( \zeta \) both falling between \( \rho(x+y) \) and \( \rho(x) \) such that

\[
|h_r(x+y) - h_r(x)| \\
\leq \phi_r(\rho_t^2(x)) \cdot |\psi(\rho_t(x+y)) - \psi(\rho_t(x))| \\
+ \psi(\rho_t(x+y)) \cdot |\phi_r(\rho_t^2(x+y)) - \phi_r(\rho_t^2(x))| \\
= \phi_r(\rho_t^2(x)) \cdot |\delta - \zeta^{-2} | \cdot |\rho_t(x+y) - \rho_t(x)| \\
+ \psi(\rho_t(x+y)) \cdot |2\zeta \phi'_r(\zeta^2)| \cdot |\rho_t(x+y) - \rho_t(x)|,
\]
which implies by Condition (1.7) that

$$|h_r(x + y) - h_r(x)| \leq \frac{\delta}{2} e^{\frac{1}{2}K} \cdot f_r(x) \cdot |\rho_t(x + y) - \rho_t(x)|$$

$$+ \frac{3e^{\frac{1}{2}K}}{N} \cdot f(x) \int_{r-K<\rho_t^2(x)<r+N+K} |\rho_t(x + y) - \rho_t(x)|.$$

Due to $\Gamma_\nu(\rho_t, \rho_t) \leq 1$, it follows

$$\Gamma_\nu(h_r, h_r) = \frac{1}{2} \int_{\mathbb{R}^n - \{0\}} |h_r(x + y) - h_r(x)|^2 \nu(x, dy)$$

$$\leq \frac{1}{2} \delta^2 e^{\delta K} f_r^2(x) + \frac{18e^{\delta K}}{N^2} \int f^2(x) \mathbf{1}_{r-K<\rho_t^2(x)<r+N+K}$$

$$\leq \frac{1}{2} \delta^2 e^{\delta K} f_r^2(x) + \frac{18e^{\delta(2N+3K)}}{N^2} e^{\delta(r-N-K)} \mathbf{1}_{r-N-K<\rho_t^2(x)<r+N+K}.$$

Let $\eta_1 = \frac{\delta}{2e^K}$ and $\eta_2 = \frac{18e^{\delta(2N+3K)}}{N^2}$, inserting the above estimate into (2.4) gives

$$\int f_r^2 d\mu \leq \eta_1 \int f_r^2 d\mu + \frac{b + ct}{c} \int h_r^2 d\mu$$

$$+ \eta_2 e^{\delta(r-N-K)} \mu\{r - N - K < \rho_t^2 < r + N + K\}.$$

**Step 3.** Choose some big $N$ and small $\delta$ so that $\eta_1 + 2\eta_2 < 1$. Since $\mu$ is a probability, there exists a sequence of $n_k \in \mathbb{N}$ such that for each $r_k = n_k(N + K)$

$$\mu\{r_k - N - K < \rho_t^2 < r_k\} \geq \mu\{r_k < \rho_t^2 < r_k + N + K\},$$

which implies

$$e^{\delta(r-N-K)} \mu\{r_k - N - K < \rho_t^2 < r_k + N + K\} \leq 2 \int f_r^2 \mathbf{1}_{r_k-N-K<\rho_t^2\leq r_k} d\mu \leq 2 \int f_r^2 d\mu.$$

It follows from Step 2

$$\int f_r^2 d\mu \leq (\eta_1 + 2\eta_2) \int f_r^2 d\mu + \frac{b + ct}{c} \int h_r^2 d\mu,$$

and thus

$$\int f_r^2 d\mu \leq \frac{b + ct}{c(1 - \eta_1 - 2\eta_2)} \int h_r^2 d\mu = C \int h_r^2 d\mu.$$

Recall $h_r = \frac{L}{m}$, fix a domain $\Omega$ with $\rho_t^2 \geq 2C$ on $\Omega^c$, which means for $r_k > \text{diam}\Omega$

$$\int f_r^2 d\mu \leq C \int_{\Omega^c} \frac{f^2}{\rho_t^2} d\mu + \frac{1}{2} \int_{\Omega^c} f_r^2 d\mu.$$

Consequently, we get $\int f^2 d\mu = \lim_{k \to \infty} \int f_r^2 d\mu \leq 2C \int_{\Omega^c} \frac{L^2}{m^2} d\mu < \infty$. \qed

### 3 Proof of Proposition 1.8

We firstly derive a Poincaré like inequality.

**Lemma 3.1.** If the Gozlan’s type condition (1.10) holds, there exist two constants $\lambda_1$ and $\lambda_2$ with big $R$ such that for any $h \in \mathcal{D}(\mathcal{E})$

$$\int h^2 d\mu \leq \lambda_1 \sum_{i=1}^{m} \frac{|h|^2}{1 + x_i^2} d\mu + \lambda_2 \int_{B(0, R+1)} h^2 d\mu.$$
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Proof. For convenience, denote $a = \frac{27}{22}$, $d\nu_i = e^{-aV}dx_i$ and

$$d\tilde{x}_i = dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_m$$

so that $d\mu = e^{-(1-a)V}d\nu_i d\tilde{x}_i$. Define $\phi_r \in C^1(\mathbb{R}^n)$ as

$$\phi_r(x) = \begin{cases} 1, & |x| \leq r; \\ 2(|x| - r)^3 - 3(|x| - r)^2 + 1, & r < |x| < r + 1; \\ 0, & |x| \geq r + 1, \end{cases}$$

which satisfies $0 \leq \phi_r \leq 1$ and $|\phi_r' | \leq 6 \frac{|x|}{|x|^2} \sqrt{1 - \phi_r}$. The proof has three steps.

**Step 1.** For any $\epsilon > 0$, there exists $R > 0$ by (1.10) such that for all $|x| \geq R$

$$\sum_{i=1}^{m} \left( a|V'_i|^2 - V''_i \right) \frac{1}{1 + x_i^4} \geq m - \epsilon.$$

It follows for any $h \in \mathcal{D}(\mathcal{E})$

$$(m - \epsilon) \int h^2 d\mu = (m - \epsilon) \int h^2 \phi_R + h^2(1 - \phi_R)d\mu$$

$$\leq (m - \epsilon) \int h^2 \phi_R d\mu + \int h^2(1 - \phi_R) \sum_{i=1}^{m} \left( a|V'_i|^2 - V''_i \right) \frac{1}{1 + x_i^4} d\mu$$

$$= (m - \epsilon) \int h^2 \phi_R d\mu + \sum_{i=1}^{m} \int \frac{h^2(1 - \phi_R) e^{-(1-a)V}}{(1 + x_i^4)} \left( a|V'_i|^2 - V''_i \right) d\nu_i d\tilde{x}_i. \quad (3.1)$$

Set $U^{(i)} = \frac{h^2(1 - \phi_R) e^{-(1-a)V}}{(1 + x_i^4)}$. For the reader’s convenience, recall the integration by parts formula satisfied by $\nu_i$, we have

$$\int U^{(i)} (a|V_i'|^2 - V''_i) d\nu_i d\tilde{x}_i = \int (U^{(i)})' V'_i d\nu_i d\tilde{x}_i$$

$$= \int \left[ 2hh'_i(1 - \phi_R) - (\phi_R)' h^2 V'_i - \frac{2x_i}{1 + x_i^4} h^2 V'_i (1 - \phi_R) - (1 - a)h^2 |V'_i|^2 (1 - \phi_R) \right] \frac{1}{1 + x_i^4} d\mu.$$ 

Using the Cauchy-Schwarz inequality gives for any positive $\epsilon_1, \epsilon_2$ and $\epsilon_3$

$$2hh'_i |V'_i| \leq \epsilon_1 h^2 |V'_i|^2 + \epsilon_2^{-1} |h'_i|^2,$$

$$-(\phi_R)' h^2 V'_i \leq 6 \frac{|x_i|}{|x|} \sqrt{1 - \phi_R} \cdot h^2 |V'_i| \leq 3\epsilon_2 h^2 |V'_i|^2 (1 - \phi_R) + 3\epsilon_2^{-1} \frac{|x_i|^2}{|x|^2} h^2,$$

$$- \frac{2x_i h^2 V'_i}{1 + x_i^4} \leq \epsilon_3 h^2 |V'_i|^2 + \frac{x_i^2 h^2}{\epsilon_3 (1 + x_i^4)^2},$$

which implies by combining the above estimates subject to $\epsilon_1 + 3\epsilon_2 + \epsilon_3 = 1 - a$

$$\int U^{(i)} (a|V'_i|^2 - V''_i) d\nu_i d\tilde{x}_i$$

$$\leq \int \frac{|h'_i|^2 (1 - \phi_R)}{\epsilon_1 (1 + x_i^4)} + \frac{3|x_i|^2 h^2}{\epsilon_2 (1 + x_i^4)|x|^2} + \frac{x_i^2 h^2 (1 - \phi_R)}{\epsilon_3 (1 + x_i^4)^3} d\mu. \quad (3.2)$$

**Step 2.** Since $\frac{x_i^2}{(1 + x_i^4)^3} \leq \frac{1}{27}$ for any $x_i$ and there exists $x_j$ with $|x_j|^2 \geq |x|^2/m$, we have

$$\sum_{i=1}^{m} \frac{x_i^2}{(1 + x_i^4)^3} \leq \frac{4}{27}(m - 1) + \frac{1}{(1 + m^{-1}|x|^2)^3}.$$
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which implies
\[
\sum_{i=1}^{m} \int \frac{x_i^2 h^2(1 - \phi_R)}{\varepsilon_3(1 + x_i^3)^3} \, d\mu \leq \left( \frac{4(m - 1)}{27\varepsilon_3} + \frac{m^2}{\varepsilon_3 R^4} \right) \int_{B(o, R)} h^2 \, d\mu. \tag{3.3}
\]

We also have
\[
\sum_{i=1}^{m} \int \frac{3|x_i|^2 h^2}{\varepsilon_2(1 + x_i^3)|x|^2} \, d\mu \leq \frac{3}{\varepsilon_2} \int_{B(o, R)} h^2 \, d\mu + \frac{3m}{\varepsilon_2 R^2} \int_{B(o, R)} h^2 \, d\mu. \tag{3.4}
\]

Choose \( R \) (depending on \( \varepsilon \) and \( \varepsilon_{1,2,3} \)) so big that \( \frac{m^2}{\varepsilon_3 R^4} + \frac{3m}{\varepsilon_2 R^2} \leq \varepsilon \), then combining (3.1-3.4) gives
\[
(m - \varepsilon) \int h^2 \, d\mu \leq \frac{1}{\varepsilon_1} \int \sum_{i=1}^{m} \frac{|h_i|^2}{1 + x_i^3} \, d\mu + \left( m - \varepsilon + \frac{3}{\varepsilon_2} \right) \int_{B(o, R+1)} h^2 \, d\mu + \left( \frac{4(m - 1)}{27\varepsilon_3} + \varepsilon \right) \int h^2 \, d\mu. \tag{3.5}
\]

**Step 3.** We have to decide the range of \( \varepsilon \) and \( \varepsilon_{1,2,3} \). First of all, fix \( \varepsilon \) such that \( \varepsilon_1 + 3\varepsilon_2 < \frac{4}{27\varepsilon_3} \), and take any \( \varepsilon_2 \) such that \( \varepsilon_1 + 3\varepsilon_2 < \frac{4(m - 1)}{27\varepsilon_3} \), too. It follows
\[
\frac{4(m - 1)}{27\varepsilon_3} = \frac{4(m - 1)}{27(1 - a - \varepsilon_1 - 3\varepsilon_2)} < m,
\]
so we can take any \( \varepsilon \) such that \( \frac{4(m - 1)}{27\varepsilon_3} + 2\varepsilon < m \).

Now, using (3.5) yields
\[
\int h^2 \, d\mu \leq \lambda_1 \int \sum_{i=1}^{m} \frac{|h_i|^2}{1 + x_i^3} \, d\mu + \lambda_2 \int_{B(o, R+1)} h^2 \, d\mu, \tag{3.6}
\]
where \( \lambda_1 = [\varepsilon_1(m - 2\varepsilon - \frac{4(m - 1)}{27\varepsilon_3})]^{-1} \) and \( \lambda_2 = (m - \varepsilon + 3\varepsilon_2^{-1})(m - 2\varepsilon - \frac{4(m - 1)}{27\varepsilon_3})^{-1} \). The proof is completed. \( \Box \)

Under the Gozlan’s condition, we use a similar argument.

**Proof of Proposition 1.8.** Let \( \beta_n = \int |x|^{2n} \, d\mu \). Applying (3.6) to \( h(x) = |x|^n \) yields
\[
\beta_n \leq \lambda_1 \int \sum_{i=1}^{m} \frac{n^2 x_i^2}{1 + x_i^3} |x|^{2n-4} \, d\mu + \lambda_2 \int_{B(o, R+1)} |x|^{2n} \, d\mu \leq \lambda_1 m n^2 \int |x|^{2n-4} \, d\mu + \lambda_2 (R + 1)^2 \int_{B(o, R+1)} |x|^{2n-2} \, d\mu \leq \lambda_1 m n^2 \beta_{n-2} + \lambda_2 (R + 1)^2 \beta_{n-1}, \tag{3.7}
\]
which implies all \( \beta_n < \infty \).

For simplicity, abbreviate \( \lambda_1' = \lambda_1 m \) and \( \lambda_2' = \lambda_2(R + 1)^2 \). Combining the Hölder inequality with (3.7) gives
\[
\beta_n = \int |x|^{n+1} |x|^{-n-1} \, d\mu \leq \beta_{n+1}^{\frac{1}{2}} \beta_{n-1}^{\frac{1}{2}} \leq [\lambda_1'(n + 1)^2 \beta_{n-1} + \lambda_2'(n + 1)^2] \beta_{n-1}^{\frac{1}{2}} \beta_{n-1}^{\frac{1}{2}},
\]
which implies
\[
\beta_n \leq \frac{\lambda_2' + \sqrt{\lambda_1'^2 + 4\lambda_1'(n + 1)^2}}{2} \beta_{n-1} \leq \left[ \lambda_2' + \sqrt{n_1'(n + 1)} \right] \beta_{n-1}.
\]

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Choose any $\gamma > \sqrt{\lambda_1}$, it follows $\lambda_2' + \sqrt{\lambda_1'(n+1)} \leq \gamma n$ for big $n$, which yields a constant $C$ such that for all $n$

\[ \beta_n \leq C \gamma^n n! . \]

By the same argument as (2.3) for any $\delta < \gamma^{-1} < \lambda_1^{-\frac{1}{2}}$, we have $\mu e^{\delta|x|^2} < \infty$.

Recall the constraints on all parameters (See Step 3 in the proof of Lemma 3.1), $\delta$ is allowed to be not greater than

\[ \sup \left\{ \lambda_1^{-\frac{1}{2}} : \epsilon_1 + 3\epsilon_2 + \epsilon_3 = 1 - a, \epsilon_1 < \frac{4}{27m}, \epsilon = \epsilon_2 = 0 \right\} , \]

which achieves $\frac{2(\sqrt{m} - \sqrt{m-1})}{3\sqrt{3m}}$. The proof is completed.

References


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