Approximating the Rosenblatt process by multiple Wiener integrals

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Abstract

Let \( Z^H \) be the Rosenblatt process with the representation
\[
Z^H_t = \int_0^t \int_0^t L^H(t, s, r) dB_s dB_r,
\]
where \( B \) is a standard Brownian motion, \( \frac{1}{2} < H < 1 \) and \( L^H \) is a given kernel. By reviewing the kernel \( L^H \) we construct its approximation of multiple Wiener integrals of the form
\[
\int_0^t \int_0^t \left\{ k_1(sr)^{-\frac{1}{2}H} + k_2(s \lor r)^{\frac{1}{2}H}(s \land r)^{-\frac{1}{2}H}|s - r|^{H-1}\right\} dB_s dB_r, \quad k_1, k_2 \geq 0.
\]

We find an optimal approximation of \( Z^H \) via calculating accurately the values of \( k_1, k_2 \).

Keywords: Rosenblatt process; optimal approximation; multiple Wiener integrals.

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1 Introduction

Hermite process is a special class of self-similar processes with long-range dependence. The processes arise from the Non Central Limit Theorem studied by Taqqu [12, 13] and Dobrushin-Majòr [7]. The famous fractional Brownian motion and Rosenblatt process are its special examples. Let us briefly recall the general context.

Let \( (\xi_n)_{n \in \mathbb{N}} \) be a stationary centered Gaussian sequence with \( E(\xi_n^2) = 1 \) such that
\[
r(n) := E(\xi_0 \xi_n) = n^{\frac{2l-2}{2l+1}} L(n),
\]
where \( l \geq 1 \) is an integer, \( H \in (\frac{1}{2}, 1) \) and \( L \) is a slowly varying function at infinity, and let the Borel function \( g : \mathbb{R} \to \mathbb{R} \) satisfy \( E(g(\xi_0)) = 0, E(g(\xi_0)^2) < \infty \) and
\[
g(x) = \sum_{j=0}^{\infty} c_j H_j(x), \quad c_j = \frac{1}{j!} E[g(\xi_0) H_j(\xi_0)],
\]

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where $H_j$ is the Hermite polynomial of order $j$ defined by

$$H_j(x) = (-1)^j e^{x^2} \frac{d^j}{dx^j} e^{-x^2}, \quad j = 1, 2, \ldots$$

with $H_0(x) = 1$. Then, the constant

$$l = \min\{j : c_j \neq 0\}.$$ 

is called the Hermite rank of $g$. Clearly, $l \geq 1$ since $E[g(\xi_0)] = 0$. For a Borel function $g$ with the Hermite rank $l$, the Non Central Limit Theorem implies that the stochastic processes

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\xi_j), \quad n = 1, 2, \ldots$$

converges, as $n \to \infty$, in the sense of finite dimensional distributions to the process

$$Z_t(H, l) = \int_{[0,t]} L^H(t, s_1, \ldots, s_l) dB_{s_1} \cdots dB_{s_l}, \quad t \in [0,1],$$

(1.2)

where $B = \{B_t, t \geq 0\}$ is a standard Brownian motion and

$$L^H(t, s_1, \ldots, s_l) = c(H, l) \left( \prod_{j=1}^{l} s_j^{\frac{1}{2} - H'} \right) \int_0^t u^{(H' - \frac{1}{2})} \prod_{j=1}^{l} (u - s_j)^{H' - \frac{3}{2}} du$$

(1.3)

with $H' = 1 - \frac{1-H}{2} \in (1 - \frac{1}{2}, 1)$, $s_1, \ldots, s_k \in [0, l]$ and a positive normal constant $c(H, l)$ such that $E[(Z_1(H, l))^2] = 1$.

**Definition 1.1** (Taqqu [13]). The process $(Z_t(H, l))_{t \geq 0}$ defined by (1.2) is called the Hermite process of order $l$ with index $H$.

Clearly, when $l = 1$ Hermite process is the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. When $l = 2$ the Hermite process is called the Rosenblatt process (see Taqqu [12]). It is important to note that Hermite process is not Gaussian for $l \geq 2$. The simplest Hermite process is fractional Brownian motion, and the Rosenblatt process is the simplest non-Gaussian Hermite process. Hermite processes are neither a semi-martingale nor a Markov process, and the following properties hold:

(i) they are the long-range dependence in the sense of

$$\sum_{n \geq 1} E[Z_1(H, l)Z_{n+1}(H, l) - Z_n(H, l)] = \infty;$$

(ii) they are $H$-selfsimilar;

(iii) they have stationary increments;

(iv) they admit the same covariance functions, i.e.

$$E[Z_t(H, l)Z_s(H, l)] = \frac{1}{2} \left[ t^{2H} + s^{2H} - |t - s|^{2H} \right];$$

(v) they are Hölder continuous of order $\gamma < H$.

These good properties of the Hermite process motivate us to study it. More works for the Hermite process and Rosenblatt process can be found in Bardet et al [3], Chen et al [4], Chronopoulou et al [5, 6], Garzón et al [8], Maejima-Tudor [9], Peccati and Taqqu [10], Pipiras-Taqqu [11], Torres-Tudor [14], Tudor [15], Tudor-Viens [16] and the references
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therein. In this paper we will prove an approximation theorem of Rosenblatt process based on the multiple integrals of form

$$\int_0^t \int_0^t \{k_1(sr)^{1-H} + k_2(s+r)^{1-H}(s \land r)^{1-H}|s-r|^{H-1}\} dB_s dB_r, \quad t \geq 0$$  \hspace{1cm} (1.4)

with \(k_1, k_2 > 0\). For simplicity we denote \(Z_t(H, 2) = Z_t^H\). The motivation to consider the approximation arises from the following estimate:

$$L^H(t, s, r) \leq C_{H,T} \left\{ \langle sr \rangle^{-H} + \langle s \lor r \rangle^{1-H}(\langle s \land r \rangle)^{1-H}|s-r|^{H-1} \right\}$$  \hspace{1cm} (1.5)

for all \(t \in [0, T]\) and \(s, r > 0\). In order to prove the above estimate, without loss of generality, we may assume that \(s \geq r\) and we have

$$\int_s^t \frac{du}{(u-s)^{1-H}(u-r)^{1-H}} = \frac{1}{(s-r)^{H-1}} \int_0^\infty \frac{dx}{x^{1-H}(1+x)^{1-H}}$$

by making the substitutions \(u - s = x(s-r)\). It follows that

$$L^H(t, s, r) = c(H, 2) \int_s^t \frac{du}{(u-s)^{1-H}(u-r)^{1-H}} + \frac{c(H, 2)}{(u-s)^{1-H}} \int_s^t \frac{du}{x^{1-H}(1+x)^{1-H}}$$

for all \(t \in [0, T]\) and \(y_1, y_2 > 0\). In general, for every Borel measurable function \(\zeta \in L_2([0, T]^2)\) the stochastic integral

$$M_t(\zeta) := \int_0^t \int_0^t \zeta(s, r) dB_s dB_r, \quad t \in [0, T]$$

is well-defined, and the best approximation problem is to estimate

$$\inf_{\zeta \in L_2([0, T]^2)} \sup_{t \in [0, T]} E(Z_t^H - M_t(\zeta))^2.$$  \hspace{1cm} (1.6)

It is important to note that if the above minimum is attained at the function \(\zeta^*\), then \(\zeta^* > 0\) a.e. In fact, we have

$$E \left(Z_t^H - M_t(\zeta)^2 \right) = t^{2H} + 2 \int_0^t \int_0^t \zeta^2(s, r)dsdr$$

$$- 4 \int_0^t \int_0^t L^H(t, s, r)\zeta(s, r)dsdr$$  \hspace{1cm} (1.7)

for all \(t \geq 0\). If \(\zeta^*(y_1, y_2) \neq 0\), then

$$\sup_{t \in [0, T]} E \left(Z_t^H - M_t(\zeta^*)^2 \right) \geq \sup_{t \in [0, T]} E \left(Z_t^H - M_t(\zeta^*)^2 \right).$$

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This gives the contradiction. Thus, we may assume that $k_1, k_2 > 0$ in (1.4) and study the best approximation problem

$$\inf_{\zeta \in \mathcal{K}} \sup_{t \in [0,T]} E(Z_t^H - M_t(\zeta))^2,$$

where

$$\mathcal{K} = \left\{ \zeta(s, r) = k_1(s r)^{-\frac{1}{2}H} + k_2(s \lor r)^{\frac{1}{2}H} (s \land r)^{-\frac{1}{2}H} | s - r|^{H-1}, \ k_1, k_2 > 0 \right\}.$$

For $\zeta \in \mathcal{K}$ we denote

$$f(t, k_1, k_2) := E(Z_t^H - M_t(\zeta))^2$$

with $t \geq 0$.

When $l = 1$, Hermite process is a fractional Brownian Motion with Hurst index $H$ and the similar approximation is first considered by Banna-Mishura [1, 2]. When $l \geq 2$, the question has not been studied and this process is non-Gaussian with non-trivial analysis.

In order to state our object, let us consider the kernel $K^H$ of the form

$$K^H(t, s) = c_H s^{-\frac{1}{2}H} \int_s^t u^{H-\frac{1}{2}} (u - s)^{H-\frac{3}{2}} du$$

where $c_H = \left( \frac{H(2H-1)}{\beta(2H-H-H)} \right)^{\frac{1}{2}}$ and $\beta(\cdot, \cdot)$ denotes the classical Beta function. Then we have (see, for example, Tudor [15])

$$L^H(t, s, r) = d(H) \int_{s \lor r}^t \frac{\partial K^H}{\partial u}(u, s) \frac{\partial K^H}{\partial u}(u, r) du,$$

where $d(H) = \frac{1}{\Gamma(1+H)} \sqrt{(4H-2)H^{-1}}$ and $H' = \frac{1}{2}(1 + H)$. In this short note, our main aim is to find the optimal approximation of $Z_t^H$ by (1.4) via calculating accurately the values of $k_1, k_2$. In order to end this one can easily check that (see Section 3)

$$\frac{\partial}{\partial t} f(t, k_1, k_2)$$

is a quadratic polynomial in $x = k_1 t^{-2H}$ and its discriminant is also a quadratic polynomial in $k_2$ with the discriminant

$$D_1 = 16 \left[ \frac{2C_2(H) 1-H}{C_1(H)(-1-H, H)} - C_1(H)^2 \right]^2 - 16 \left[ \beta^2(1-H, H) - \frac{\beta(1-H, 2H-1)}{1-H} \right] \left[ C_1(H)^2 - \frac{2H}{1-H} \right]$$

for $H \in (\frac{1}{2}, 1)$, where $C_1(H) = d(H)c_{H'}^{2H}\beta^2(1-H, \frac{1}{2}H)$ and

$$C_2(H) = d(H)c_{H'}^{2H} \int_0^1 \int_0^s r^{-H}(1-s)^{-\frac{1}{2}H-1}(1-r)^{\frac{1}{2}H-1}(s-r)^{H-1} dr ds.$$

By using the constant $D_1$ we give our main result and at the end of this paper we give the numerical simulations of these constants (see Figure 1, 2, 3 and Table 1).

This note is organized as follows. In Section 2, we give the representation of the function $f(t, k_1, k_2) = E(Z_t^H - M_t(\zeta))^2$ for $\zeta \in \mathcal{K}$. In Section 3 and Section 4, we consider the optimal approximation in the two cases $D_1 \leq 0$ and $D_1 > 0$, respectively. In Section 5 we consider two special cases.
2 The representation of $f(t, k_1, k_2)$

In order to give the representation of $f(t, k_1, k_2) = E \left( Z_t^H - M_t(\zeta) \right)^2$ for $\zeta \in K$, we start with the finiteness of the constant $C_2(H)$.

**Lemma 2.1.** For all $\frac{1}{2} < H < 1$ the integral

$$\int_0^1 \int_0^s r^{-H} (1-s)^{\frac{1}{2}H-1}(1-r)^{\frac{1}{2}H-1}(s-r)^{H-1}drds$$

converges.

**Proof.** By Young’s inequality, we have

$$(1-r)^{\frac{1}{2}H-1} \leq (1-s)^{(\frac{1}{2}H-1)\gamma}(s-r)^{(\frac{1}{2}H-1)(1-\gamma)}$$

for all $0 < \gamma < 1$. Notice that $1 - \frac{3}{2}H < \frac{1}{2}H$ for all $\frac{1}{2} < H < 1$. We get

$$\int_0^1 \int_0^s r^{-H} (1-s)^{\frac{1}{2}H-1}(1-r)^{\frac{1}{2}H-1}(s-r)^{H-1}drds$$

$$\leq \int_0^1 (1-s)^{(\frac{1}{2}H-1)(1+\gamma)}dy_1 \int_0^s r^{-H}(s-r)^{\frac{3}{2}H-2+(1-\frac{1}{2}H)\gamma}dy_2$$

$$= \int_0^1 s^{-(1-\gamma)(1-\frac{1}{2}H)}(1-s)^{-(1-\frac{1}{2}H)(1+\gamma)}ds \int_0^1 x^{-H}(1-x)^{\frac{3}{2}H-2+(1-\frac{1}{2}H)\gamma}dx < \infty,$$

for all $\frac{1-\frac{3}{2}H}{1-\frac{1}{2}H} \leq 0 < \frac{\frac{3}{2}H}{1-\frac{1}{2}H}$. This proves $C_2(H) < \infty$. \hfill \Box

**Theorem 2.2.** Let $C_1(H)$ and $C_2(H)$ be given in Section 1. Denote

$$a(k_2) := 1 + H^{-1}2(k_2)^2\beta(1 - H, 2H - 1) - 4k_2H^{-1}C_2(H)$$

and

$$b(k_2) := C_1(H) - 2k_2\beta(1 - H, H)$$

for all $k_1, k_2 \geq 0$ and $\frac{1}{2} < H < 1$. Then we have

$$f(t, k_1, k_2) = a(k_2)t^{2H} - 4k_1b(k_2)t + \frac{2k_2^2}{(1-H)^2}t^{2-2H}, \quad t \in [0, T].$$

As an immediate result we see that $a(k_2) \geq 0$ and

$$a(k_2) - 2(1-H)^2b^2(k_2) > 0$$

(2.1)

for all $k_2$ since $f(t, k_1, k_2) \geq 0$ is a quadratic equation in $k_1$. Notice that $a(k_2)$ is also a quadratic equation in $k_2$. We get

$$2(C_2(H))^2 \leq H\beta(1 - H, 2H - 1)$$

for all $\frac{1}{2} < H < 1$.

**Proof of Theorem 2.2.** An elementary calculation can show that
\[
\int_0^t \int_0^r L^H(t, s, r)(sr)^{-\frac{1}{2}H} ds dr \\
= d(H)c^2_H \int_0^t \int_0^r (sr)^{-H} u^H(u - s)^{\frac{1}{2}H - 1} (u - r)^{\frac{1}{2}H - 1} dud s r \\
= \int_0^t \int_0^r d(H)c^2_H (sr)^{-H} u^H(u - s)^{\frac{1}{2}H - 1} (u - r)^{\frac{1}{2}H - 1} ds dr du \\
= d(H)c^2_H \int_0^t \int_0^r s^{-H} (1 - s)^{\frac{1}{2}H - 1} r^{-H} (1 - r)^{\frac{1}{2}H - 1} ds dr du \\
= d(H)c^2_H \beta^2 (1 - H, \frac{1}{2}H) t = C_1(H)t
\]
and
\[
\int_0^t \int_0^r L^H(t, s, r)(s \lor r)^{\frac{1}{2}H} (s \land r)^{-\frac{1}{2}H} (s - r)^{H - 1} ds dr \\
= \int_0^t \int_0^r d(H)c^2_H (sr)^{-\frac{1}{2}H} u^H(u - s)^{\frac{1}{2}H - 1} (u - r)^{\frac{1}{2}H - 1} \\
\cdot (s \lor r)^{\frac{1}{2}H} (s \land r)^{-\frac{1}{2}H} (s - r)^{H - 1} ds dr du \\
= d(H)c^2_H \int_0^t \int_0^r u^{2H - 1} (sr)^{-\frac{1}{2}H} (1 - s)^{\frac{1}{2}H - 1} (1 - r)^{\frac{1}{2}H - 1} \\
\cdot (s \lor r)^{\frac{1}{2}H} (s \land r)^{-\frac{1}{2}H} (s - r)^{H - 1} ds dr du \\
= H^{-1} d(H)c^2_H \int_0^t \int_0^r r^{-H} (1 - s)^{\frac{1}{2}H - 1} (1 - r)^{\frac{1}{2}H - 1} (s - r)^{H - 1} ds dr \\
= C_2(H) H^{-1} \beta^{2H} \\
\]
for all \( t \in [0, T] \), which give
\[
\int_0^t \int_0^r L^H(t, s, r) \zeta(s, r) ds dr = k_1 C_1(H)t + k_2 C_2(H) H^{-1} \beta^{2H}
\]
for all \( t \in [0, T] \). On the other hand, it is easy to calculate that
\[
\int_0^t \int_0^r \zeta^2(s, r) ds dr = \frac{k_1^2}{(1 - H)^2} t^{2 - 2H} + \frac{k_2^2}{H} \beta (1 - H, 2H - 1) t^{2H} + 4k_1 k_2 \beta (1 - H, H) t
\]
for all \( \zeta \in \mathcal{K} \). It follows that
\[
f(t, k_1, k_2) = E \left( Z^H_t - M_t(\zeta) \right)^2 \\
= t^{2H} + 2 \int_0^t \int_0^r \zeta^2(s, r) ds dr - 4 \int_0^t \int_0^r L^H(t, s, r) \zeta(s, r) ds dr \\
= a(k_2) t^{2H} - 4k_1 b(k_2) t + \frac{2k_1^2}{(1 - H)^2} t^{2 - 2H}
\]
for all \( \zeta \in \mathcal{K} \). This completes the proof. \( \square \)

### 3 The optimal approximation, case \( D_1 \leq 0 \)

In order to obtain the optimal approximation in the case \( D_1 \leq 0 \) we need some preliminaries and keep the notation in Section 2. Denote \( \alpha = H - \frac{1}{2} \) and define the quadratic function \( x \mapsto G(x) \) on \([0, \infty)\) by
\[
G(x) := \frac{2}{1 - H} x^2 - 2b(k_2) x + H a(k_2)
\]
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for \( x \geq 0 \). Consider the maximum of \( t \mapsto f(\cdot, \cdot, t) \). We have for all \( t, k_1, k_2 > 0 \),

\[
\frac{\partial}{\partial t} f(t, k_1, k_2) = 2H a(k_2) t^{2H-1} - 4k_1 b(k_2) + \frac{4k_2^2}{1-H} t^{1-2H}
\]
\[
= t^{2\alpha} \left( \frac{4k_2^2}{1-H} t^{-4\alpha} - 4b(k_2)k_1 t^{-2\alpha} + 2H a(k_2) \right)
\]
\[
= 2\alpha G(x)
\]

with \( x = k_1 t^{-2\alpha} \). Clearly, the discriminant \( D \) of the quadratic polynomial \( G(x) \) satisfies

\[
\frac{1}{4} D = (b(k_2))^2 - \frac{2H}{1-H} a(k_2) = 4 \left[ \beta^2 (1-H, H) - \frac{\beta(1-H, 2H-1)}{1-H} \right] k_2^2 + 4 C(H) - C(H) \beta(1-H, H)
\]
\[
= 2 \left[ 2C(H) - C(H) \beta(1-H, H) \right] k_2 + C(H)^2 - \frac{2H}{1-H}.
\]

This gives a quadratic polynomial in \( k_2 \) and its discriminant is \( D_1 \).

**Theorem 3.1.** If \( D_1 \leq 0 \), then we have

\[
\inf_{\zeta} \sup_{0 \leq t \leq T} E \left( Z^H_t - M_t(\zeta) \right)^2 = a(k_2)^2 T^{2H} - 4k_1 b(k_2) T + \frac{2k_2^2}{1-H^2} T^{2-2H}
\]

where

\[
\zeta(s, r) = k_1^* (s \lor r)^{\alpha} + k_2^* (s \land r)^{\alpha} |s - r|^{\alpha-1}, \quad s, r > 0
\]

and \((k_1^*, k_2^*)\) is the stagnation point of the function

\[
(k_1, k_2) \mapsto f(T, k_1, k_2).
\]

An elementary calculation can obtain

\[
k_1^* = \frac{2(1-H)^2 \beta(1-H, H) C_2(H) - (1-H)^2 \beta(1-H, 2H-1) C_1(H)}{4H(1-H)^2 \beta(1-H, H) - \beta(1-H, 2H-1) T^{2\alpha}}
\]
\[
k_2^* = \frac{2H(1-H)^2 \beta(1-H, H) C_1(H) - C_2(H)}{4H(1-H)^2 \beta(1-H, H) - \beta(1-H, 2H-1)}
\]

**Lemma 3.2.** For all \( \frac{1}{2} < H < 1 \) we have \( \beta^2 (1-H, H) - \frac{\beta(1-H, 2H-1)}{1-H} < 0 \).

**Proof.** This is a simple exercise. In fact, for all \( \frac{1}{2} < H < 1 \) we have

\[
\beta^2 (1-H, H) = \left( \int_0^1 x^{-H} (1-x)^{H-1} x^{-H} dx \right)^2
\]
\[
\leq \int_0^1 \left( x^{-H} (1-x)^{H-1} \right)^2 dx \int_0^1 x^{-H} dx = \frac{\beta(1-H, 2H-1)}{1-H}
\]

by Cauchy inequality, and it is easy to check that the inequality above is strict. \( \square \)

**Proof of Theorem 3.1.** Let now \( D_1 \leq 0 \). Then we see that \( D \leq 0 \) and \( \frac{\partial f}{\partial t} \geq 0 \) for \( H \in (\frac{1}{2}, 1) \) and \( k_1, k_2 \geq 0 \). It follows that

\[
\sup_{0 \leq t \leq T} E \left( Z^H_t - M_t(\zeta) \right)^2 = f(T, k_1, k_2)
\]
\[
= a(k_2) T^{2H} - 4k_1 b(k_2) T + \frac{2k_2^2}{(1-H)^2} T^{2-2H}
\]

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for all $k_1, k_2 \geq 0$. Let now $(k^*_1, k^*_2)$ be the stagnation point of the function $(k_1, k_2) \mapsto f(T, k_1, k_2)$.

Then $(k^*_1, k^*_2)$ can be given by (3.3) and (3.4), and elementary calculations may obtain the Hessian matrix $H$ on $f(T, k_1, k_2)$ as follows

$$
H = \left( \begin{array}{cc} \frac{\partial^2 f(T, k_1, k_2)}{\partial k_1^2} & \frac{\partial^2 f(T, k_1, k_2)}{\partial k_1 \partial k_2} \\ \frac{\partial^2 f(T, k_1, k_2)}{\partial k_2 \partial k_1} & \frac{\partial^2 f(T, k_1, k_2)}{\partial k_2^2} \end{array} \right) = \left( \begin{array}{cc} \frac{4T^{2-2H}}{(1-H)^2} & 8\beta(1-H)T \\ 8\beta(1-H)T & \frac{4}{\beta} \beta(1-H)^2 - 4\beta(1-H) \end{array} \right)
$$

and $|H| = 16T^2 \left( \frac{(1-H^2H-1)}{H(1-H)} - 4\beta(1-H, H) \right)$. Combining this with (3.5), we get $|H| > 0$ for all $H \in (\frac{1}{2}, 1)$, which means that the minimal value of $(k_1, k_2) \mapsto f(T, k_1, k_2)$ is achieved at the point $(k^*_1, k^*_2)$. Thus, we have proved the theorem.

4 The optimal approximation, case $D_1 > 0$

In this section we throughout let $D_1 > 0$ and keep the notation in Section 3 and Section 2. When $D_1 > 0$, the equation $D = 0$ admits two real roots as follows

$$
k_{2,1} = -4 \left[ \frac{2C_4(H)}{1-H} - C_4(H)\beta(1-H, H) \right]^{\frac{1}{2}} \frac{\sqrt{D_1}}{8 \left[ \beta^2(1-H, H) - \frac{(1-H)2H-1}{1-H} \right]}
$$

and

$$
k_{2,2} = -4 \left[ \frac{2C_4(H)}{1-H} - C_4(H)\beta(1-H, H) \right] - \sqrt{D_1} \frac{8 \left[ \beta^2(1-H, H) - \frac{(1-H)2H-1}{1-H} \right]}{8 \left[ \beta^2(1-H, H) - \frac{(1-H)2H-1}{1-H} \right]}
$$

We get

$$
\inf_{\zeta \in K} \sup_{0 \leq t \leq T} E \left( Z_t^H - M_t(\zeta) \right)^2 = \inf_{\zeta \in K} \sup_{0 \leq t \leq T} f(t, k_1, k_2)
$$

$$
= \min \left\{ \inf_{k_2 \in \{k_{2,2}, k_{2,1}\}} \sup_{0 \leq t \leq T} f(t, k_1, k_2), \inf_{k_1^* > 0} \sup_{0 \leq t \leq T} f(t, k_1, k_2) \right\}.
$$

Thus, we can complete the discussion in two cases: $k^*_2 \notin \{k_{2,2}, k_{2,1}\}$ and $k^*_2 \in \{k_{2,2}, k_{2,1}\}$.

**Theorem 4.1.** If $D_1 > 0$ and $k^*_2 \notin \{k_{2,2}, k_{2,1}\}$, then we have

$$
\inf_{\zeta \in K} \sup_{0 \leq t \leq T} E \left( Z_t^H - M_t(\zeta) \right)^2 = a(k^*_2)T^{2H} - 4k^*_1b(k^*_2)T + \frac{2k^*_1}{(1-H)^2} T^{2-2H},
$$

where

$$
\zeta(y_1, y_2) = k^*_1 (y_1 y_2)^{-\alpha} + k^*_2 (y_1 \vee y_2)^{\alpha} (y_1 \wedge y_2)^{-\alpha} |y_1 - y_2|^{2\alpha - 1}, \quad y_1, y_2 > 0.
$$

**Proof.** Let $k^*_2 \notin \{k_{2,2}, k_{2,1}\}$. By (4.1) we have that

$$
\inf_{k_2 \in \{k_{2,2}, k_{2,1}\}} \sup_{0 \leq t \leq T} f(t, k_1, k_2) = f(T, k^*_1, k^*_2),
$$

provided $D \leq 0$, and

$$
\inf_{k_2 \in \{k_{2,2}, k_{2,1}\}} \sup_{0 \leq t \leq T} f(t, k_1, k_2) \geq \inf_{k_2 \in \{k_{2,2}, k_{2,1}\}} f(T, k_1, k_2) > f(T, k^*_1, k^*_2),
$$

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provided $D > 0$, which imply
\[ \inf_{\zeta \in \mathcal{K}} \sup_{0 \leq t \leq T} f(t, k_1, k_2) = f(T, k_1^*, k_2^*), \]
and the theorem follows. \hfill \Box

Next we consider the case $k_2^* \in (k_{2,1}, k_{2,1})$.

**Lemma 4.2.** For $k_2^* \in (k_{2,1}, k_{2,1})$, we have
\[ \inf_{\zeta \in \mathcal{K}} \sup_{0 \leq t \leq T} f(t, k_1, k_2) = \inf_{k_2 \in (k_{2,2}, k_{2,1})} \sup_{0 \leq t \leq T} f(t, k_1, k_2).\]

**Proof.** By (4.1) it is enough to show that
\[ \inf_{k_2 \in (k_{2,1}, k_{2,1})} \sup_{0 \leq t \leq T} f(t, k_1, k_2) \leq \inf_{k_2 \in (k_{2,1}, k_{2,1})} \sup_{0 \leq t \leq T} f(t, k_1, k_2). \tag{4.2} \]

If $k_2 \not\in (k_{2,1}, k_{2,1})$, then $D \leq 0$ and we have
\[ \sup_{0 \leq t \leq T} f(t, k_1, k_2) = f(T, k_1, k_2). \]

By solving the equation $\frac{\partial f(T, k_2, k_1)}{\partial t} = 0$, we get the stagnation points of the functions $k_1 \mapsto f(T, k_1, k_2)$ and $k_1 \mapsto f(T, k_1, k_{2,1})$ as follow
\[ k_{1,1} := [C_1(H) - 2k_{2,1} \beta(1 - H, H)] (1 - H)^2 T^{2\alpha} \]
and
\[ k_{1,2} := [C_1(H) - 2k_{2,1} \beta(1 - H, H)] (1 - H)^2 T^{2\alpha}, \]
respectively. It follows that
\[ \inf_{k_2 \in (k_{2,1}, k_{2,1})} \sup_{0 \leq t \leq T} f(t, k_1, k_2) = \min \{f(T, k_1, k_{2,1}), f(T, k_1, k_{2,1})\}. \]

Clearly, if $k_2 = k_{2,2}$ or $k_2 = k_{2,1}$, we have $D = 0$. So $\frac{\partial f}{\partial t} \geq 0$ and
\[ \sup_{0 \leq t \leq T} f(t, k_1, k_{2,1}) = f(T, k_1, k_{2,1}). \]

Hence we have
\[ \inf_{k_2 \in (k_{2,1}, k_{2,1})} \sup_{0 \leq t \leq T} f(t, k_1, k_2) \leq \inf_{k_1 > 0} \sup_{0 \leq t \leq T} f(t, k_1, k_2) \]
\[ = \inf_{k_1 > 0} f(T, k_1, k_{2,1}) = f(T, k_1, k_{2,1}). \]

On the other hand, we can also get
\[ \inf_{k_2 \in (k_{2,1}, k_{2,1})} \sup_{0 \leq t \leq T} f(t, k_1, k_2) \leq f(T, k_1, k_{2,1}), \]
and the inequality (4.2) follows. This completes the proof. \hfill \Box

Clearly, $D > 0$ if $k_2 \in (k_{2,1}, k_{2,1})$, and by (3.1) we can see that the equation
\[ \frac{\partial f}{\partial t} = 2t^{2\alpha} \left( \frac{2}{1 - H} x^2 - 2b(k_2)x + Ha(k_2) \right) = 0 \tag{4.3} \]
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has two real roots as follows

\[ x_1 := \frac{1 - H}{2} \left( b(k_2) + \left( b^2(k_2) - \frac{2Ha(k_2)}{1 - H} \right)^{1/2} \right) \]

and

\[ x_2 := \frac{1 - H}{2} \left( b(k_2) - \left( b^2(k_2) - \frac{2Ha(k_2)}{1 - H} \right)^{1/2} \right), \]

which says \( t_1 := t_1(k_1, k_2) = k_1^{\frac{1}{n}} x_1^{\frac{1}{n}} \) and \( t_2 := t_2(k_1, k_2) = k_1^{\frac{1}{n}} x_2^{\frac{1}{n}} \) are the two stagnation points of the function \( t \mapsto f(t, k_1, k_2) \). It follows from the monotonicity of the function \( t \mapsto f(t, k_1, k_2) \) that \( t_1 := t_1(k_1, k_2) \) and \( t_2 := t_2(k_1, k_2) \) are the points of local maximum and minimum, respectively, which implies that

\[
\sup_{t \in [0,T]} f(t, k_1, k_2) = \begin{cases} 
    f(T, k_1, k_2), & t_1 \geq T \\
    \max \{ f(t_1, k_1, k_2), f(T, k_1, k_2) \}, & t_1 < T 
\end{cases}
\]

and

\[
f(t_1, k_1, k_2) = a(k_2) \left( \frac{k_1}{x_1} \right)^{\frac{H}{n}} - 4k_1 b(k_2) \left( \frac{k_1}{x_1} \right)^{\frac{1}{n}} + \frac{2k_1^2}{(1 - H)^2} \left( \frac{k_1}{x_1} \right)^{\frac{1-n}{n}}.
\]

\[
(4.4)
\]

**Lemma 4.3.** If \( k_2^* \in (k_{2,2}, k_{2,1}) \), we then have

\[ t_1(k_1^*, k_2^*) < T. \]

**Proof.** Noting that \( t_1 = k_1^{\frac{1}{n}} x_1^{\frac{1}{n}} \) and \( t_2 = k_1^{\frac{1}{n}} x_2^{\frac{1}{n}} \), we get \( t_1^{2\alpha} = \frac{x_1}{k_1}, t_2^{2\alpha} = \frac{x_2}{k_1} \) and

\[
\frac{2}{t_1^{2\alpha}} + \frac{1}{t_2^{2\alpha}} = \frac{x_1 + x_2}{k_1} = \frac{(1 - H)b(k_2)}{k_1}
\]

since \( t_1 < t_2 \). When \( k_1 = k_1^* \) and \( k_2 = k_2^* \), we have

\[
\frac{(1 - H)b(k_2)}{k_1^*} = \frac{(1 - H)(C_1(H) - 2k_2^* \beta(1 - H, H))}{\xi(H)T^{2\alpha}} = \frac{1}{(1 - H)T^{2\alpha}} > \frac{2}{T^{2\alpha}}.
\]

\[
(4.5)
\]

for \( H \in \left(\frac{1}{2}, 1\right) \), where

\[
\eta(H) := \frac{2H(1 - H)^2 \beta(1 - H, H)C_1(H) - C_2(H)}{4H(1 - H)^2 \beta^2(1 - H, H) - \beta(1 - H, 2H - 1)}
\]

and

\[
\xi(H) := \frac{2(1 - H)^2 \beta(1 - H, H)C_2(H) - (1 - H)^2 \beta(1 - H, 2H - 1)C_1(H)}{4H(1 - H)^2 \beta^2(1 - H, H) - \beta(1 - H, 2H - 1)}.
\]

This proves that \( t_1(k_1^*, k_2^*) < T. \)

**Lemma 4.4.** If \( k_2^* \in (k_{2,2}, k_{2,1}) \), we then have \( t_2(k_1^*, k_2^*) < T \).

**Proof.** From (4.5) it follows that

\[
k_1^* = (1 - H)^2 T^{2\alpha} b(k_2^*).
\]

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On the other hand, (3.1) implies that

\[
\frac{\partial f(t, k_1^*, k_2^*)}{\partial t} = 2^{2\alpha} \left( \frac{2(k_1^*)^2}{1-H}(t^{-2\alpha})^2 - 2b(k_2^*)k_1^*t^{-2\alpha} + Ha(k_2^*) \right) = g(t^{-2\alpha})
\] (4.7)

is a quadratic function in \( x = t^{-2\alpha} \), and

\[ g(t_1^{-2\alpha}(k_1^*, k_2^*)) = g(t_2^{-2\alpha}(k_1^*, k_2^*)) = 0. \]

Noting that

\[
g(T^{-2\alpha}) = 2T^{2\alpha} \left( \frac{2(k_1^*)^2}{1-H}(T^{-2\alpha})^2 - 2b(k_2^*)k_1^*T^{-2\alpha} + Ha(k_2^*) \right)
\]

\[
= 2T^{2\alpha} \left( \frac{2(1-H)^2T^{2\alpha}b(k_2^*)^2}{1-H}(T^{-2\alpha})^2 - 2b(k_2^*)(1-H)^2T^{2\alpha}b(k_2^*)T^{-2\alpha} + Ha(k_2^*) \right)
\]

\[
= 2HT^{2\alpha} \left[ a(k_2^*) - 2(1-H)^2b^2(k_2^*) \right] > 0
\]

by (2.1), we get

\[ T^{-2\alpha} \notin [t_2^{-2\alpha}(k_1^*, k_2^*), t_1^{-2\alpha}(k_1^*, k_2^*)] \]

and \( t_2(k_1^*, k_2^*) < T \) by a simple analysis and Lemma 4.3. \( \Box \)

**Lemma 4.5.** Denote

\[ h(k_1, k_2) = f(t_1, k_1, k_2) - f(T, k_1, k_2). \]

For any \( k_2 \in (k_{2,1}, k_{2,2}) \), the equation \( h(k_1, k_2) = 0 \) (with unknown \( k_1 \)) has two solutions \( k_{1-} \) and \( k_{1+} \), which satisfy \( 0 < k_{1-} < k_{1+} < 0 \) and \( \frac{\partial h}{\partial k_1} \big|_{k_1=\hat{k}_1} > 0 \) and \( \frac{\partial h}{\partial k_1} \big|_{k_1=\bar{k}_1} = 0 \).

**Proof.** Clearly, \( h(k_1, k_2) = 0 \) and (4.4) imply that

\[
k_1^{\frac{2-H}{2H-1}} \varphi(k_2) = a(k_2)T^{2H} - 4k_1b(k_2)T + \frac{2k_1^2}{(1-H)^2}T^{2-2H} = f(T, k_1, k_2)
\] (4.9)

for all \( k_2 \in (k_{2,2}, k_{2,1}) \). Differentiating (4.9) with respect to \( k_1 \) and multiplying by \( \frac{2H}{2H-1}k_1 \) on both sides of the equation (4.9) lead to

\[
\frac{2H}{2H-1}k_1^{\frac{2-H}{2H-1}} \varphi(k_2) = -4b(k_2)T + \frac{4k_1}{(1-H)^2}T^{2-2H}
\] (4.10)

and

\[
\frac{2H}{2H-1}k_1^{\frac{2-H}{2H-1}} \varphi(k_2) = \frac{2H}{(2H-1)k_1} a(k_2) T^{2H}
\]

\[
- \frac{8H}{(2H-1)} b(k_2) T + \frac{4Hk_1}{(2H-1)(1-H)^2} T^{2-2H}
\] (4.11)

for all \( k_2 \in (k_{2,2}, k_{2,1}) \). It follows that

\[
-4b(k_2)T + \frac{4k_1}{(1-H)^2}T^{2-2H} = \frac{2H}{(2H-1)k_1} a(k_2) T^{2H}
\]

\[
- \frac{8H}{(2H-1)} b(k_2) T + \frac{4Hk_1}{(2H-1)(1-H)^2} T^{2-2H},
\]

which implies that

\[
Ha(k_2) T^{2H} - 2b(k_2)Tk_1 + \frac{2}{1-H} T^{2-2H} k_1^2 = 0
\] (4.12)
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for all \( k_2 \in (k_{2,1}, k_{2,1}). \) This is a quadratic equation in \( k_1 \) with the two roots

\[
\bar{k}_1 = \frac{1 - H}{2} T^{2\alpha} (b(k_2) + \sqrt{D}) = x_1 T^{2\alpha}, \quad \underline{k}_1 = \frac{1 - H}{2} T^{2\alpha} (b(k_2) - \sqrt{D}) = x_2 T^{2\alpha}
\]

because its discriminant

\[
\Delta = 4T^2 \left[ b^2(k_2) - \frac{2H}{1 - H} a(k_2) \right] = T^2 D > 0.
\]

It is easily to check that \( \bar{k}_1 \) is the solution to the equation

\[
h(k_1, k_2) = 0 \tag{4.13}
\]

and \( \frac{\partial h}{\partial x_1} |_{k_1 = \bar{k}_1} = 0 \) for all \( k_2 \in (k_{2,2}, k_{2,1}). \) In order to see that \( \bar{k}_1 \) is not the solution to the equation (4.13), we claim that \( h(k_1, k_2) \neq 0 \) for all \( k_2 \in (k_{2,2}, k_{2,1}). \) We have

\[
h(k_1, k_2) = f(t_1, \bar{k}_1, k_2) - f(T, \bar{k}_1, k_2)
\]

\[
= x_2^2 \frac{\varphi(k_2)}{\varphi(k_1)} T^{2H} - a(k_2) T^{2H} + 4x_2 b(k_2) T^{2H} - \frac{2x_2^2}{(1 - H)^2} T^{2H}
\]

\[
= T^{2H} \left[ \left( \frac{x_2}{x_1} \right)^{\frac{2H}{H-1}} \left( a(k_2) - 4b(k_2)x_1 + \frac{2x_1^2}{(1 - H)^2} \right) - \left( a(k_2) - 4b(k_2)x_2 + \frac{2x_2^2}{(1 - H)^2} \right) \right].
\]

Put \( u = x_1 \) and \( z = x_2, \) then \( u \) and \( z \) are the roots of the equation (4.3), and by (3.1) we have

\[
\frac{2u^2}{(1 - H)^2} = \frac{2b(k_2)u}{1 - H} - Ha(k_2) \quad \text{and} \quad \frac{2z^2}{(1 - H)^2} = \frac{2b(k_2)z}{1 - H} - Ha(k_2)
\]

and

\[
u + z = (1 - H)b(k_2), \quad uz = \frac{H(1 - H)}{2} a(k_2). \tag{4.14}
\]

It follows that

\[
h(k_1, k_2) = \frac{2H - 1}{1 - H} T^{2H} \left[ \left( \frac{z}{u} \right)^{\frac{2H}{H-1}} (2b(k_2)u - a(k_2)) - 2b(k_2)z + a(k_2) \right]
\]

for all \( k_2 \in (k_{2,2}, k_{2,1}). \) Now, we claim that the inequality

\[
\left( \frac{z}{u} \right)^{\frac{2H}{H-1}} (2b(k_2)u - a(k_2)) - 2b(k_2)z + a(k_2) > 0 \tag{4.15}
\]

for all \( k_2 \in (k_{2,2}, k_{2,1}). \) According to (4.14), the above inequality is equivalent to

\[
a(k_2) \left( \left( \frac{z}{u} \right)^{\frac{2H}{H-1}} \left( H(u + z) \right)^2 - 1 \right) - \left( \frac{H(u + z)}{u} - 1 \right) > 0 \tag{4.16}
\]

for all \( k_2 \in (k_{2,2}, k_{2,1}), \) and it can be simplified as

\[
\phi(x) := Hx^{\frac{2H}{H-1}} - (1 - H)x^{\frac{2H}{H-1}} + (1 - H) - Hx > 0
\]

with \( x = \frac{z}{u} \in (0, 1). \) This is a simple calculus exercise. In fact, we have \( \phi(0) = 1 - H, \)

\[
\phi(1) = 0,
\]

\[
\phi'(x) = \frac{H}{2H - 1} x^{\frac{2H}{H-1} - 1} - \frac{2H(1 - H)}{2H - 1} x^{\frac{2H}{H-1} - 1} - H
\]
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and

\[ \phi''(x) = \frac{2H(1 - H)}{(2H - 1)^2} \frac{z^{2H}}{\sin(\pi H)} - \frac{2H(1 - H)}{(2H - 1)^2} \frac{z^{2H}}{\sin(\pi H)} > 0 \]

for all \( x \in (0, 1) \) since \( 2H > 1 \). This shows that the function \( \phi \) is convex on \( (0, 1) \) and \( \phi' \) is increasing strictly on \( (0, 1) \), which gives

\[ -H = \phi'(0) < \phi'(x) < \phi'(1) = 0 \]

for \( x \in (0, 1) \). It follows that \( \phi \) is strictly decreasing on \( (0, 1) \) and

\[ \phi(x) > \phi(1) = 0 \]

for \( x \in (0, 1) \). Thus, we have showed that the inequality (4.15) holds and \( h(k_1, k_2) > 0 \) for all \( k_2 \in (k_{2,2}, k_{2,1}) \).

On the other hand, from \( h(0, k_2) = -a(k_2)T^{2H} < 0 \) it follows that the equation

\[ h(k_1, k_2) = f(t_1, k_1, k_2) - f(T, k_1, k_2) = 0 \]

admits a root, denoted by \( \hat{k}_1 \), on \( (0, \xi_1) \) for all \( k_2 \in (k_{2,2}, k_{2,1}) \). Noting that the function \( k_1 \mapsto f(t_1, k_1, k_2) \) is convex and increasing, we find easily that the equation

\[ h(k_1, k_2) = f(t_1, k_1, k_2) - f(T, k_1, k_2) = 0 \]

admits two roots at most since the function \( k_1 \mapsto f(T, k_1, k_2) \) is a quadratic function. Thus, \( \hat{k}_1 \) is unique in \((0, \xi_1)\) and \( \frac{\partial h}{\partial k_1} \big|_{k_1=\hat{k}_1} > 0 \), and the lemma follows.

Now, we can give the solution of the second case.

**Theorem 4.6.** Let \( D_1 > 0 \) and \( k^*_2 \in (k_{2,2}, k_{2,1}) \). Suppose that \( \hat{k}_1 \) and \( t_1 \) are given as above. Then there exists \( \hat{k}_2 \in [k_{2,2}, k_{2,1}] \) such that the minimal value

\[ \inf_{\zeta \in K} \sup_{0 \leq t \leq T} f(t, k_1, k_2) \]

is achieved at the point \((T, \hat{k}_1, \hat{k}_2)\) and this value equals to \( f(T, \hat{k}_1, \hat{k}_2) \).

**Proof.** Let \( D_1 > 0 \) and \( k^*_2 \in (k_{2,2}, k_{2,1}) \). Then, Lemma 4.3 and Lemma 4.4 imply that

\[ \inf_{\zeta \in K} \sup_{0 \leq t \leq T} f(t, k_1, k_2) = \inf_{k_2 \in (k_{2,2}, k_{2,1})} \max_{k_1 \geq 0} \{ f(t_1, k_1, k_2), f(T, k_1, k_2) \} \]

It follows from Lemma 4.5 that

\[ \max\{ f(t_1, k_1, k_2), f(T, k_1, k_2) \} = f(t_1, k_1, k_2) 1_{\{ k_1 > \hat{k}_1 \}} + f(T, k_1, k_2) 1_{\{ k_1 < \hat{k}_1 \}} \]

which implies that

\[ \max\{ f(t_1, k_1, k_2), f(T, k_1, k_2) \} = f(T, \hat{k}_1, k_2) \]

because \( k_1 \mapsto f(t_1, k_1, k_2) \) is increasing and \( f(T, k_1, k_2) \) is decreasing for \( k_1 < \hat{k}_1 \). Combining this with the continuity of \( k_2 \mapsto f(T, k_1, k_2) \), we see that there exists \( \hat{k}_2 \in [k_{2,2}, k_{2,1}] \)

such that

\[ \inf_{k_2 \in (k_{2,2}, k_{2,1})} f(T, \hat{k}_1, k_2) = f(T, \hat{k}_1, \hat{k}_2) \]

This completes the proof.

**5 Two special cases**

In this section we consider two special classes of the approximation functions \( \zeta \in K \).

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Let $K_1 = \left\{ \zeta(s, r) = k(sr)^{-\frac{1}{2}H}, \ k > 0 \right\}$.

(1) If $(C_1(H))^2 \leq \frac{2H}{1-H}$. Then

$$\inf_{\zeta \in K_1} \sup_{0 \leq t \leq T} E \left( Z_t^H - M_t(\zeta) \right)^2 = \left[ 1 - 2(1 - H)^2 C_1(H)^2 \right] T^{2H}$$

with $\zeta(s, r) = (1 - H)^2 C_1(H) T^{2\alpha}(sr)^{-\frac{1}{2}H}$, $s, r > 0$.

(2) If $(C_1(H))^2 > \frac{2H}{1-H}$. Then

$$\min_{\zeta \in K_1} \sup_{0 \leq t \leq T} E \left( Z_t^H - M_t(\zeta) \right)^2 = f(T, k^*, 0),$$

where $\zeta(s, r) = \hat{k}(sr)^{-\frac{1}{2}H}$ ($s, r > 0$) and $\hat{k}$ is the smallest root of the equation $f(T, k, 0) - f(t_1, k, 0) = 0$.

Proof. For $\zeta \in K_1$ we have

$$E \left( Z_t^H - M_t(\zeta) \right)^2 = f(t, k, 0) = t^{2H} - 4kC_1(H)t + \frac{2k^2}{(1 - H)^2} t^{2-2H}$$

and $D = 4 \left( C_1(H)^2 - \frac{2H}{1-H} \right)$, which complete the proof. \(\square\)

Finally, denote $K_2 = \left\{ \zeta(s, r) = k(s \vee r)^{\frac{1}{2}H}(s \wedge r)^{-\frac{1}{2}H}|s - r|^{H-1}, \ k > 0 \right\}$. Then, for $\zeta \in K_2$, by (2.2) and $a(k) > 0$ we have

$$\inf_{\zeta \in K_2} \sup_{t \in [0, T]} E \left( Z_t^H - M_t(\zeta) \right)^2 = \inf_{\zeta \in K_2} a(k) T^{2H}$$

$$= a(k^*) T^{2H} = T^{2H} - \frac{2(C_2(H))^2}{1-H, 2H-1} T^{2H}$$

with $k^* = \frac{C_2(H)}{\beta(1 - H, 2H - 1)}$ and

$$\zeta(s, r) = \frac{C_2(H)}{\beta(1 - H, 2H - 1)} (s \vee r)^{\frac{1}{2}H}(s \wedge r)^{-\frac{1}{2}H}|s - r|^{H-1}, \ s, r > 0.$$
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Figure 1: The function $H \mapsto C_1(H)$

Figure 2: The function $H \mapsto C_2(H)$

Table 1: The enumeration of some constants

<table>
<thead>
<tr>
<th>H</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
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<td>$C_1(H)$</td>
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<td>2.0581</td>
<td>3.3500</td>
<td>6.9514</td>
</tr>
<tr>
<td>$C_2(H)$</td>
<td>1.1220</td>
<td>1.1113</td>
<td>1.0777</td>
<td>0.8969</td>
</tr>
<tr>
<td>$D_1(H)$</td>
<td>-106.9256</td>
<td>-8.0811</td>
<td>853.7535</td>
<td>$4.463 \times 10^4$</td>
</tr>
</tbody>
</table>

Figure 3: The function $H \mapsto D_1(H)$